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AN UNIFIED APPROACH FOR THE CONTRACTIVE MAPPINGS

By

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ABSTRACT

In this paper we introduce a new definition of contractive type mappings called p-contraction. A selfmapping T of metric space (X, d) is called a p-contraction if there exists a map $p : R^3{}_* - R_*$ where $R_* = \{t \in R : t \geqslant 0\}$, satisfying some suitable hypothesis, such that,

$$d(Tx, Ty) \leqslant p(\, d(x, y), d(x, Tx), d(y, Ty)\,).$$

for all pair $x, y \in X$.
The definition of p-contraction, which contains some of previous generalization of contraction mappings, implies all conclusions of Banach contraction principle.

1. INTRODUCTION

We recall Banach's contraction principle and its basic consequences. A selfmapping of a metric spac (X, d) is called a contraction if there exists h, $0 \leqslant h < 1$. such that for each $x, y \in X$

$$d(Tx, Ty) \leqslant h\, d(x, y).$$

Any contraction map T of a complete metric space (X, d) has the following properties :

(A) T has a unique fixed point $z_* \in X$.

(B) Convergence of iterates $T^n x \to z_*$, as $n \to \infty$.

(C) Uniform convergence : there exists a neighbourhood U of z_* such that $T^n(U) = \{T^n x : x \in U\} \to \{z_*\}$, which means that for any neighbourhood V of z_* we can find an integer n_0 such that $T^n(U) \subseteq V$ for all n greater than n_0.

(D) T is continuous.

On the other hand, Meyers [6] proved that every selfmapping T of complete metric space (X, d) having properties A, B, C and D is a contraction map under suitable topologically equivalent metric.

The purpose of this paper is to investigate p-contraction mappings (see precise definition below), mappings having some iterate which is a p-contraction and common fixed points of two mappings.

Applications of previous theorem of Meyers are also given.

2. Results on p-Contraction Mappings.

Let us introduce the set P of continuous $p : R^3_* \to R_*$, where $R_* = \{t \in R : t \geqslant 0\}$ satisfying the following properties :

(i) $p(1, 1, 1) = h < 1$;

(ii) Let $u > 0$. If $u \leqslant p(u, v, v)$ or $u \leqslant p(v, u, v)$ or $u \leqslant p(v, v, u)$ then $u \leqslant hv$.

Definition 1. A selfmapping of a metric space (X, d) is called a p-contraction if there exists a map $p \in P$, such that, for all $x, y \in X$

$$d(Tx, Ty) \leqslant p(\, d(x, y), d(x, Tx), d(y, Ty))$$

Now we indicate 3 examples of contractive type mappings, which can be regarded as a p-contraction, under suitable function p.

Indeed, If we put $p(u, v, z) = au + bv + cz$ $(a + b + c < 1)$, then we get the following well known :

Definition 2. (S. Reich [7], I. Rus [9]).

There exist non negative numbers a, b, c, $a + b + c < 1$, such that for all pair $x, y \in X$

$d(Tx, Ty) \leqslant a\, d(x, y) + b\, d(x, Tx) + c\, d(y, Ty)$.

Moreover if we put $p(u, v, z) = h \max (u, v, z)$ $(0 \leqslant h < 1)$, we have

Definition 3.

There exists a number h, $0 \leqslant h < 1$, such that for all pair $x, y \in X$

$$d(Tx, Ty) \leqslant h \max \{d(x, y), d(x, Tx), d(y, Ty)\},$$

At last if we choose $p(u, v, z) = (au^r + bv^r + cz^r)^{1/r}$, $r > 0$, $(a + b + c < 1)$ then we find the following :

Definition 4. (Delbosco) [2].

There exist non negative numbers a, b, c and there exists $r > 0$, such that for all pair $x, y \in X$

$$d(Tx, Ty)^r \leqslant a\, d(x, y)^r + b\, d(x, Tx)^r + c\, d(y, Ty)^r.$$

Theorem 2.1 Let T be a p-contraction mapping of a complete metric space (X, d). Then T has properties (A) and (B).

Proof. Fix $z \in X$. Suppose $z \neq Tz$ and set $z_n = T z_{n-1} = T^n z$. if we put $x = z_n$ and $y = z_{n-1}$, then inequality (1) becomes :

(2.1) $d(z_{n+1}, z_n) \leqslant p(\ d(z_n, z_{n-1}), d(z_n, z_{n+1}), d(z_{n-1}, z_n)\ ).$

Hypothesis (ii) on function p implies

(2.2) $d(z_{n+1}\ , z_n) \leqslant h\ d(z_n\ , z_{n-1}).$

On the other hand, hypothesis (i) on p, proves that $(z_n)_{n \epsilon N}$ is a Cauchy sequence and completeness of X assures that $z_n \to z_* \in X$.

To prove that z_* is fixed point of T, we take $x = z_*$ and $y = z_n$ in inequality (1) and we obtain

(2.3) $d(Tz_*, Tz_n) \leqslant p(\ d(z_*, z_n), d(z_*, Tz_*), d(z_n, Tz_n)\ ).$

Since p is continuous in each variable, letting $n \to \infty$, we get

(2.4) $d(Tz_*, z_*) \leqslant p(\ 0, d(z_*, Tz_*), 0).$

$d(z_*, Tz_*)$ must be zero; hence $z_* = Tz_*$.

Finally we prove that mapping T has a unique fixed point.

Indeed, let us suppose by contradiction, that $Tx = x$, $Ty = y$ and that $x \neq y$. Erom inequality (1) we get :

$d(x, y) = d(Tx, Ty) \leqslant p(\ d(x, y), 0, 0).$

It follows that $d(x, y) = 0$ *i. e.* $x = y$.

The proof is, thus, complete

Theorem 2.2 Let T be a continuous p-contraction selfmapping of a complete metric space (X, d). Then T has property (C).

Proof. For any $r > 0$, we define $V_r = \{x \in X : d(x, Tx) < r\}$. We show that V_r is an open subset of X, since it contains a neighbourhood for any point. Indeed, if $y \in V_r$, $d(y, Ty) = a < r$, then continuity of T implies that there exists r such that

(2.5) $d(z, y) < r'$ implies $d(z, Tz) < \frac{a+r}{2} < r.$

V_r is a neighbourhood of unique fixed point z_*, since $z_* \in V_r$ and V_r is an open subset of mearic space X.

Now we prove property (C) about T, i e.

(2.6) $\{T^n V_r\} \to \{z_*\}$, as n becomes infinite.

Let us consider $z = Ty \in T(V_r) = \{Tx : x \in V_r\}$. From inequality (1) we can write that

(2.7) $d(Ty, T^2y) \leqslant p(\, d(y, Ty), d(y, Ty), d(Ty, T^2y)\,)$

we use hypothesis (ii) on function to deduce that

(2.8) $d(Ty, T^2y) \leqslant h\, d(y, Ty).$

Therefore it follows that

(2.9) $T(V_r) \subseteq V_{hr}$

and, by induction,

(2.10) $T^n(V_r) \subseteq V_{h^n r}$.

Thus, inclusion (2.10) proves (2.6) (uniform convergence of iterates – or – property (C)).

Theorem 2.3 Let T be a continuous p-contraction selfmapping of a complete metric space (X, d). Then T is a contraction map for suitable topologically equivalent metric.

Proof. It suffices to remark that T has properties (A), (B), (C) from previous Theorem 2.1 and 2.2 and T has also property (D) for hypothesis. Now applying Meyers's result [6], one proves this theorem.

3. MAPPINGS HAVING SOME ITERATE p-CONTRACTION.

Sometimes a selfmapping T is not a p-contraction, but it can have some iterate which is a p-contraction mapping, under suitable function $p \in P$.

From this point of view, we state and prove some results.

Theorem 3.1 Let T be a selfmapping of complete metric space (X, d) satisfying the following condition

(1*) there exists an integer n such that for all pair $x, y \in X$

$$d(T^n x, T^n y) \leqslant p(\, d(x, y), d(x, T^n x), d(y, T^n y)\,)$$

for some $p \in P$. Then T has properties (A) and (B).

Proof. If we put $S = T^n$,. then Theorem 2.1 shows that S has a unique fixed point that we call z_*.

Now we prove that z_* is also a fixed point of T. Indeed let us consider these equalities

(3.1) $T^n(Tz_*) = T(T^n z_*)$

$T^n(Tz_*) = Tz_*$

$S(Tz_*) = Tz_*$

Uniqueness of fixed point of S and (3.1) imply

(3.2) $Tz_* = z_*$

Now we prove that z_* is the unique fixed point of T. Suppose $x \neq y$, $Tx = x$ and $Ty = y$ and apply (1*)

(3.3) $d(x, y) = d(T^n x, T^n y) \leqslant p(\, d(x, y), 0, 0)$

$$d(x, y) \leqslant p(0, 0, d(x, y))$$

$$d(x, y) = 0$$

Hence $x = y$ and uniqueness of fixed point is proved.

To prove property (B) it suffices to remark that we can deduce the convergence of sequence $T^h z$ from the convergence of sequence $S^i z_0$, putting $z_0 = z, Tz, T^2z, \ldots, T^{n-1}z$.

Theorem 3.2 Let T a continuous selfmapping of a complete metric space (X, d) satisfying (1*). Then T has property (C).

Proof. For any $r > 0$ we define $V_r = \{x \in X : d(x, T^n x) < r\}$, where n is the integer of condition (1*).

We shall prove that V_r is an open subset of X containing the fixed point z_*, hence a neighbourhood of z_*.

Indeed if $y \in V_r$, $d(y, T^n y) = a < r$, then continuity of T implies that there exists r' such that

(3.4) $d(z, y) < r'$ implies $d(z, Tz) < \dfrac{a + r}{2} < r$.

To prove property (C), we consider sequence $\{T^{kn}(V_r)\}$ $(k \in N)$ and deduce the following convergence result

(3.5) $\{T^{kn}(V_r)_{k \in N} \longrightarrow \{z_*\}$, as k becomes infinite.

Suppose $z \in T^n(V_r)$, then we can write $z = T^n y$, with $y \in V_r$.
From condition (1*) it follows

(3.6) $d(z, T^n z) = d(T^n y, T^{2n} y) \leqslant p(d(y, T^n y), d(y, T^n y), d(T^n y, T^{2n} y))$

Now hypothesis (i) and (ii) on function p assure that

(3.7) $d(z, T^n z) \leqslant h\, d(y, T^n y) \leqslant hr$.

Therefore we have

(3.8) $T^n (V_r) \subseteq V_{hr}$

and, by induction,

(3.9) $T^{kn} (V_r) \subseteq V_{h^k r}$.

Thus uniform convergence (3.6) is proved as consequence of inclusion (3.9), letting $n \to \infty$.

4. Common Fixed Points Of Two Mappings.

The purpose of this section is to investigate common fixed points of two mappings f and g satisfying the following condition

(1**) There exists a function $p \in P$, such that, for all pair $x, y \in X$

$$d(fx, gy) \leqslant p(\, d(x, y), d(x, fx)\; d(y, gy)\,)$$

Theorem 4.1 Let f and g be two selfmapping of a complete metric space (X, d) satisfying (1**). Then f and g have a unique common fixed point.

Proof. Fix $x \in X$ and define the following sequence

(4.1) $x_1 = fx, x_2 = g\, x_1 ,\dots, x_{2n-1} = fx_{2n-2}, x_{2n} = g\, x_{2n-1} ,\dots$

If we put $y = x_1$ in (1**) we have

(4.2) $d(x_1, x_2) = d(fx, gx_1) \leqslant p(\, d(x, x_1), d(x, x_1).\, d(x_1, x_2)\,)$

$$d(x_1, x_2) \leqslant h\, d(x, x_1).$$

Similarly we get

(4.3) $d(x_2, x_3) \leqslant h\, d(x_1, x_2) \leqslant h^2\, d(x, x_1)$

and, by induction,

(4 4) $d(x_n, x_{n+1}) \leqslant h^n \, d(x, x_1)$.

Inequality (4.4) proves that sequence (4.1) is Cauchy.

Completeness of X assures that limit z_* of sequence (4.1) belongs to X.

To prove that z_* is a fixed point of f and g, we set $x = z_*$ and $y = x_{2n-1}$ in condition (1**) and deduce

(4.5) $d(fz_*, x_{2n}) \leqslant p(\, d(z_*, x_{2n-1}), d(z_*, fz_*), d(x_{2n-1}, x_{2n})\,)$

Letting $n \to \infty$, (4.5) shwos that z_* is a fixed point of g. On the other hand, z_* is the unique common fixed point of two mappings f and g. This has simple and standard proof.

Finally we make use of this Theorem 4.1, to study the class of selfmapping of a metric space (X, d) which satisfy the following condition

(1***) There exist two integers n, m, such that for all pair $x, y \in X$
$d(T^n x, T^m y) \leqslant p(\, d(x, y), d(x, T^n x), d(y, T^m y)\,)$
for some function $p \in P$.

Theorem 4.2 Let T be a selfmapping of a complete metric space (X, d) satisfying (1***). Then T has properties (A) and (B).

Proof. The proof of property (A) follows directly from Theorem 4 1 taking $f = T^n$ and $g = T^m$. Indeed there exists a unique fixed point such that $T^n z_* = z_* = T^m z_*$.

But we have

$$T^n(Tz_*) = T(T^n z_*) = Tz_* = T(T^m z_*) = T^m(Tz_*)$$

hence, uniqueness of common fixed point implies

$T(z_*) = z_*$.

To prove property (B), let us suppose that $n > m$. Theorem (4.1) implies that for any fixed $z \in X$ the sequence

(4 6) $z, T^n z, T^{n+m} z, T^{2n+m} z, T^{2n+2m} z, \ldots\ldots$

converges to fixed point z_*.

Putting $z = x, Tx, T^2x, \ldots, T^{n-1}x$, in (4 6) we obtain n sequences containing all elements of sequence $(T^k x)_{k \in N}$.

The property (B) is thus proved.

5. Comparison with some recent similar results.

Recently, some authors [3] [5] have established fixed point theorems involving function q: $R^5_* \to R_*$ satisfying :

(a) q is continuous;

(b) q is increasing, or nondecreasing; in each variable;

(c) $q(t, t, t, t, t) < t$ for each $t > 0$.

Now we present the corresponding contractive definition: A selfmapping T of a complete metric space is "q" contraction if there exists a function q such that for all pair $x, y \in X$

$$d(Tx, Ty) \leqslant p(\, d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\,).$$

Our purpose is to show that function p is not the restriction of some function q to its first three component.

Let us consider the following example :

$$p(u, v, z) = h \max \{u \exp [(u-v)(v-z)(z-u)]^2, v, z\},\ 0 \leqslant h < 1.$$

Since function p, as defined above, is not increasing or not decreasing in its first coordinate variable, p can not satisfy (b), and hence, can not be the restriction of some function q to its first three variable.

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FIXED POINTS OF SET-VALUED MAPPINGS OF CONVEX METRIC SPACES*

By

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ABSTRACT

The principal aim of the paper is to prove the existence of fixed points for set-valued mappings of convex metric spaces. Such spaces include Banach spaces and starshaped subsets of Banach spaces. and our results generalize those of Assad, Kirk, Markin, Nadler, and others.

INTRODUCTION

In 1970 Takahashi [10] introduced the definition of convexity in metric space given in Definition 0. 1 below, and generalized some

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important fixed-point theorems previously proved for Banach spaces. Subsequently. Machado [5], Tallman [11], Naimpally and Singh [8], and Naimpally, Singh and Whitfield [9], among others have obtained additional results in this setting. This paper is a continuation of these investigations for *set-valued* mappings. Our results are motivated by the results Samanta [12] and extends his results to convex metric spaces.

Definition 0. 1 Let X be a metric space and I be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x, y \in X, \lambda \in I$,
$d(u, W(x, y, \lambda)) \leqslant \lambda\, d(u, x) + (1 - \lambda)\, d(u, y)$ for all $u \in X$. The space X together with a convex structure is called a *convex metric space*. Clearly any convex subset of a Banach space is a convex metric space with $W(x, y, \lambda) = \lambda x + (1-\lambda)y$.

Definition 0. 2 Let X be a convex metric space. A nonempty subset K of X is *convex* if $W(x, y, \lambda) \in K$ whenever $x, y \in K$ and $\lambda \in I$.

Takahashi has shown that open and closed balls are convex, and that the arbitrary intersection of convex sets is convex.

Definition 0. 3 A convex metric space X will be said to have Property (C) if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

By Smulian's theorem [2, *p*. 433], every weakly compact convex subset of a Banach space has Property (C).

1 Fixed points for set-valued mappings.

Fixed points of set-valued mappings on Banach spaces have been studied extensively. In this section we present results which generalize results of Assad and Kirk [1], Kirk [3], Morkin [6], Nadler [7], and others.

A few additional definitions will be needed.

Definition 1. 1 Let X be a convex metric space and C be a nonempty, closed, convex, bounded set in X. Following Kirk [3], for $x \in X$ we set

$$r_x(C) = \sup_{y \in C} d(x, y),$$

$$r(C) = \inf \{r_x(C) : x \in C\}$$

We then define $D = \{x \in C : r_x(C) = r(C)\}$ to the *center* of C.

Definition 1. 2 If $A, B \in 2^X$, then the Hausdorff distance $D(A, B)$ is defined as $D(A, B) = \inf \{\varepsilon : A \subseteq N(B, \varepsilon)$ and $B \subseteq N(A, \varepsilon)$, where $N(C, \varepsilon) = \{x \in X : \exists\, y \in C > d(x, y) < \varepsilon\}$.

We denote the *diameter* of a subset E of X by $\delta(E) = \sup \{d(x, y) : x, y \in E\}$.

Definition 1. 3 A point $x \in E$ is a *diametral point* of E provided $\sup \{d(x, y) : y \in E\} = \delta(E)$, the diameter of E.

Definition 1. 4 A convex metric space is siad to have *normal structure* if for each closed bounded convex subset E of X which contains at least two points, there exists $x \in E$ which is not a diametral point of E.

It is clear that any compact convex metric space, and every bounded, closed convex subset of a uniformly convex Banach space will have normal structure.

Theorem 1.5 Let X be a complete convex metric space with Property (C) and K be a closed, convex, bounded subset of X with normal structure.

]

If $T : K \to 2^K$ is a mapping such that

(i) $T(x) \cap K \neq \emptyset$ for $x \in K$,

and

(ii) for every closed convex subset of L of K satisfying

$T(z) \cap K \neq \emptyset$ for all $z \in L$

$D(T(x) \cap L, T(y) \cap L) \leqslant d(x, y)$ whenever $x, y \in L$. $x \neq y$,

then T has a fixed point.

Proof. Let G be the family of all nonempty, closed convex subsets M of K such that $T(x) \cap M \neq \emptyset$ for all $x \in M$. Then G is nonempty since $K \in G$. Partially order G by inclusion. Let $\tilde{t} = \{F_i\}_{i \in \triangle}$ be a decreasing chain in G. From Property (C) if follows that $\bigcap_{i \in \triangle} F_i \neq \emptyset$. Let $F_0 = \bigcap_{i \in \triangle} F_i$, a nonempty, closed convex subset of M. Let $x \in F_0$.

By hypothesis, $T(x) \bigcap_{i \in \triangle} F_i \neq \emptyset$. For any $F_i \in \tilde{t}$, $T(x) \cap F_i$ is compact and $\{T(x) \cap F_j\}_{j \geqslant i}$ is a family of nonempty closed subsets of the compact set $T(x) \cap F_i$ having the finite intersection property. Consequently $\bigcap_{j \geqslant i} (T(x) \cap F_j) \neq \emptyset$ and therefore $\bigcap_{i \in \triangle} (T(x) \cap F_i) \neq \phi$, *i.e.* $T(x) \cap F_0 \neq \phi$. Thus any chain in G has a greatest lower bound and by Zorn's Lemma there is a minimal member N in G. We claim that N is a singleton set. If not, then as shown by Takahashi [10], the center of N, denoted by P is a nonempty *proper* closed convex subset of N. It is enough to show that P belongs to G. Let $y \in P$. Then $y \in N$ and $T(y) \cap N \neq \phi$. Take $z \in T(y) \cap N$. Let $Q = B[z, r(N)] \cap N$ (where $B_r(x) = \{y \in X : d(x, y) \leqslant r\}$). Then Q is a nonempty closed convex subset of N. Let $x \in Q$. Using hypothesis (ii) we have

$$D(T(x) \cap N, T(y) \cap N) \leqslant d(x, y).$$

Since $z \in T(y) \cap N$, there is a $u \in T(y) \cap N$ such that $d(u, y) \leqslant D(T(x) \cap N, T(y) \cap N) \leqslant d(x, y)$. Thus $u \in Q$. Herce $T(x) \cap Q \neq \phi$ for all $x \in Q$. By the minimality of N we have $N \subseteq Q$; that is, $N \subseteq B[z, r(N)]$. Consequently, $z \in P$ and we have $T(y) \cap P \neq \phi$ for all $y \in P$. Therefore $P \in G$, a contradiction of the minimality of N. Hence N consists of a single element which is a fixed point of Γ.

The following example show that condition (ii), can not be replaced by $D(Tx, Ty) \leqslant d(x, y)$ for all $x, y \in K$, $x \neq y$.

Example 1. 6 Let $X = R^2$ with the Euclidean norm and $K = \{ (x, 0) : 0 \leqslant x \leqslant 1\}$. Then K is a bounded, closed convex subset of a reflexive Banach space X. Define $T: K \to 2^X$ as follows :

$$T(x, 0) = \{(\sqrt{\tfrac{1}{2}}\,(\tfrac{1}{2} - x), \sqrt{\tfrac{1}{2}}\,(\tfrac{1}{2} - x)), (x + \tfrac{1}{2}, 0)\}, \; 0 \leqslant x \leqslant \tfrac{1}{2}$$

$$T(x + \tfrac{1}{2}, 0) = \{ (1 - \frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}), (x, 0) \}, \; 0 \leqslant x \leqslant \tfrac{1}{2}$$

Then $T(x)$ intersects K for every x in K.

If $(x, 0), (y, 0) \in K$ such that $x, y \in [0, \frac{1}{2}]$ or $x, y \in [\frac{1}{2}, 1]$, then

$$D(T(x, 0), T(y, 0)) \leqslant \| (x, 0) - (y, 0) \| .$$

If $x, y \in [0, \frac{1}{2}]$, then

$$D(T(x, 0), T(y + \tfrac{1}{2}, 0) \leqslant [(\sqrt{\tfrac{1}{2}}\,(\tfrac{1}{2} - x) - y)^2 + (\sqrt{\tfrac{1}{2}}\,(\tfrac{1}{2} - x))^2]^{1/2}$$

$$\leqslant [x^2 + y^2 + \sqrt{2}xy - (x + \sqrt{\tfrac{1}{2}}\, y) + \tfrac{1}{4}]^{1/2}$$

$$\leqslant \| (x, 0) - (y + \tfrac{1}{2}, 0) \|$$

whenever $(\sqrt{2} + 2)\, xy \leqslant (1 + \sqrt{\frac{1}{2}})\, y$; i. e., $x \leqslant \frac{1}{2}$ with $y \neq 0$.

Thus $D(\, T(x, 0), T(y, 0)) \leqslant \| (x, 0) - (y, 0) \|$ for all $(x, 0), (y, 0), x \neq y$ in K. Now for $x, y \in [0, \frac{1}{2}]$,

$$D(\, T(x, 0) \cap K, T(y + \tfrac{1}{2}. 0) \cap K)$$

$$= \| (x + \tfrac{1}{2}. 0) - (y, 0) \|$$

$$= \| x + \tfrac{1}{2} - y \|$$

$$= \| (x, 0) - (y + \tfrac{1}{2}, 0) \|$$

$$= \| y + \tfrac{1}{2} - x \|$$

whenever $x > y$. So T is nonexpansive and satisfies all but condition (ii) of Theorem 1 5. Clearly T has no fixed point.

Corollary 1. 8 Let X be a reflexive Banach space and K be a bounded, closed convex subset of X which has normal structure. Let $T: K \rightarrow 2^X$ be a mapping such that

(i) $T(x) \cap K \neq \phi$ for all $z \in K$

(ii) for any closed convex subset L of K satisfying

$$T(z) \cap L \neq \phi \text{ for all } z \in L$$

$$D(\, T(x) \cap L, T(y) \cap L) \leqslant d(x, y)$$

whenever $x, y \in L, x \neq y$. Then T has a fixed point

The following result of Takahashi (*Thm* 3 [10]) follows from Theorem 1. 5.

Corollary 1. 9 Let X be a convex metric space having Property (C). Let K be a nonempty bounded closed convex subset of X with normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

The following result of Kirk [4] follows from Corollary 1. 9.

Corollary 1. 10 Let K be a nonempty bounded, closed convex subset of a reflexive Banach space X and suppose that K has normal structure. If T is a nonexpansive mapping of K into itself, then T has a fixed point in K.

To obtain the next theorem we first establish a lemma whose proof requires that the mapping $W: X \times X \times I \to X$ be continuous in $\lambda \in I$.

Lemma 1. 11 Let H and K be two closed subsets of the convex metric space X such that $H \cap K \neq \phi$. If $W: X \times X \times I \to X$ is continuous in $\lambda \in I$ and H is convex, then $d_H(K) = \phi$ iff $H \subseteq K$. Here $d_H(K)$ denotes the realative boundary of $H \cap K$ in H.

Proof. If $H \subseteq K$, then $H - K = \phi$ and $d_H(K) = \phi$. Conversely, suppose that $d_H(K) = \phi$, while $H - K \neq \phi$. Let $x \in H \cap K$ and $y \in H - K$. Since H is convex, $W(x, y, \lambda) \in H$ for all $x, y \in H$ and $\lambda \in I$. Let $\gamma = \inf \{\lambda: W(x, y, \lambda) \in K\}$ and $\{\gamma_n\}$ be a sequence in I converging to γ such that $W(x, y, \gamma_n) \in K$ Since K is closed, and W is continuous in λ, $W(x, y, \gamma) = \lim\limits_{n} W(x, y\ \gamma_n)$ is in K. Assume that $\gamma > 0$ and let $\varepsilon > 0$ be given. Take $n \in (0,\gamma)$ such that $\gamma - n < \dfrac{\varepsilon}{d(x, y)}$. Then $W(x, y, \gamma) \in S_{\varepsilon}$. $(W(x, y, \gamma) \cap (H - K)$. It follows that $W(x, y, \gamma) \in d_H(K)$ and consequently, that $d_H(K) \neq \phi$, a contradiction. The result follows.

Theorem 1. 12 Let X be a convex metric space having Property (C) such that the mapping W is continuous in $\lambda \in I$. Let K be a nonempty closed convex subset of X. Let L be a nonempty closed bounded convex subset of K having normal structure, and $T: L \to 2^X$ be a mapping satisfying $T(x) \cap K \neq \phi$ for all $x \in L$. If M is a nonempty closed convex subset of K such that $T(x) \cap M \neq \phi$ whenever $x \in M \cap L$, and

(i) $D(T(x) \cap M, T(y) \cap M) \leqslant d(x, y)$

for all $x, y \in M \cap L$ and

(ii) $T(w) \cap (M \cap L) \neq \phi$, $w \in d_M(L)$, then T has a fixed point.

Proof. Let G be the family of all nonempty closed convex subsets M of K such that $M \cap L \neq \phi$ and $T(x) \cap M \neq \phi$ whenever $x \in M \cap L$. Then G is nonempty since $K \in G$. Partially order G be set inclusion. Let $\tilde{t} = \{F_i\}_{i \in \triangle}$ be a decreasing chain in G. Since each F_i is closed and convex, it follows from property (C) that $F_0 = \bigcap\limits_{i \in \triangle} F_i$ is a nonempty closed convex subset of M. Let $x \in F_0$. Then, by hypothesis, $T(x) \cap F_i \neq \phi$ for all $i \in \triangle$. Moreover, for any $F_i \in \tilde{t}$, $T(x) \cap F_i$ is compact and $\{T(x) \cap F_j\}_{j \geqslant i}$ is a family of nonempty closed sub ets of the compact set $T(x) \cap F_i$ having the finite intersection property. Therefore, $\bigcap\limits_{j \geqslant i} (T(x) \cap F_j) \neq \phi$ and consequently $\bigcap\limits_{i \in \triangle} (T(x) \cap F_i) \neq \phi$; that is, $T(x) \cap F_0 \neq \phi$. Thus every chain in G has a greatest lower bound and by Zorn's Lemma there is a minimal element N in G. If $d_N(L) = \phi$, then by lemma 11. 1, $N \subseteq L$. In this case Theorem 1.5 is applicable and T has a fixed point. Thus. we assume that $d_N(L) \neq \phi$ and show that $N \cap L$ is a singleton set. Let $\delta(N \cap L) > 0$. Since L has normal structure, there is a point a in $L \cap N$ such that

$$0 < a = \sup_{b \in N \cap L} \{d(a, b)\} < \delta(N \cap L) \tag{1}$$

Consider the set $H = \{x \in N : N \cap L \subseteq B[x, \alpha]\}$, a closed convex subset of N. Since $a \in H$, H is nonempty and, by (1) $H \neq N$. We show that $H \in G$. Let $z \in H \cap L$ and $y \in d_N(L)$. Then $y \in N \cap L$ and hence $d(z, y) \leqslant \alpha$. Now, since $N \in G$, from hypothesis (i) we have for $y, z \in N \cap L$:

$$D(T(y) \cap N, T(z) \cap N) \leqslant d(y, z). \tag{2}$$

Consequently, $D(T(y) \cap N, T(z) \cap N) \leqslant \alpha$.

Since $y \in d_N(L)$, by hypothesis (ii), $T(y) \cap (N \cap L) \neq \phi$. Take $u \in T(y) \cap (N \cap L)$. Then, using (2) we obtain for $v \in T(z) \cap N$, $d(u, v) \leqslant D(T(z) \cap N, T(y) \cap N) \leqslant d(z, y)$. Let $W = B[v, \alpha] \cap N$. Then W is a closed convex subset of K with $W \cap L \neq \phi$ (since $u \in W \cap L$). If $w \in W \cap L \subseteq N \cap L$, then $d(z, w) \leqslant \alpha$.

Further, $D(T(w) \cap N, T(z) \cap N) \leqslant d(z, w) \leqslant \alpha$. Since $v \in T(z) \cap N$, there exists an element s in $T(w) \cap N$ such that $d(v, s) \leqslant D(T(w) \cap N, T(z) \cap N) \leqslant d(w, z) \leqslant \alpha$. This shows that $s \in W$, or that $T(w) \cap W \neq \phi$ for all $w \in W \cap L$. Therefore $W \in G$. By the minimality of N we have $N = W$. Hence $N \cap L \subseteq N B[v, \alpha]$ and $v \in H$. We now have $T(z) \cap H \neq \phi$ for all $z \in H \cap L$ and it follows that $H \in G$. But H is a proper subset of N, a contradiction. Hence $N \cap L$ is a singleton set. Let $N \cap L = \{x\}$. Since by assumption, $\phi \neq d_N(L) \subseteq N \cap N$, $d_N(L) = \{x\}$. Now by hypothesis (ii), $x \in T(x)$. This completes the proof.

Remark 1.13 The example below shows that condition (ii) of Theorem 1 12 can not be replaced by the following condition: (ii)$'$ If $x \in d_K(L)$ then $T(x) \cap L \neq \phi$.

Example 1. 14 Let $X = R^2$ with the usual metric and $K = \{(x, 0): 0 \leqslant x \leqslant \infty\}$ and $L = \{(x, 0): 0 \leqslant x \leqslant 1\}$. Define $T: L \to 2^X$ by $T(x, 0) = \{(1 + x, 0), (0, 1 - x)\}$, $(x, 0) \in L$.

Then K is a closed and convex subset of a reflexive Banach space and L is a closed, bounded convex subset of K. The mapping T satisfies the condition $T(x) \cap K \neq \phi$ for all $x \in L$, condition (i) of Theorem 1.12, and condition (ii). By taking $M = [\frac{1}{2}, \infty)$ we see that hypothesis (ii) of Theorem 1 12 is not satisfied. Clearly T does not have a fixed point.

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COMMON FIXED POINT IN METRIC SPACES

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INTRODUCTION

In 1970, Takahashi [8] introduced a notion of convexity in metric spaces (see Definition 1 . 1) and generalized some fixed-point theorems in Banach spaces. Subsequently, Itoh [4] , Guay , Singh , and Whitfield [3] Naimpally, Singh, and whitfield [7] , Tallman [9] , and Machado [6] , among others , have studied fixed - point theorems in convex metric spaces. This paper is a continuation of these investigations.

1. Preliminaries and definition

Throughout this paper, we will assume that X is a convex metric

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space having property (C) .

Let K be a nonempty, closed, bounded convex subset of a convex metric space X . Let $A\ (K)$ denote the collection of all nonempty subsets of K, and for A , $B \in A\ (K)$, let $\delta(A, B) =$ sup $\{d(a, b) : a \in A, b \in B\}$, *let* $r_x(K) = \delta(x, k) = sup\ \{b\,(x, y) : y \in K\}$, $\delta(K)$ denotes the diameter of K. Let $r(K) — \inf(\delta(x, K) : x \in K)$, and $K_c — (x \in K : r(K) — \delta(x, K)\}$.

It is well known [8] that if K is a closed, bounded, convex subset of a convex metric space having property (C) then K_c is a nonempty closed convex subset of K . Furthermore, if K has normal structure then $\delta(K_c) < \delta(K)$, whenever $\delta(K) > 0$.

In the sequel, we use I for the interval [0, 1] . The following notion was introduced by Takahashi [8] .

Definition 1. 1 Let X be a metric space. A mapping $W: X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x, y \in X, \lambda \in I, d(u, W(x. y, \lambda)) \leqslant \lambda d(u, x) + (1 - \lambda)\, d(u, y)$ for all $u \in X$. X together with a convex structure is called a *convex metric space.*

Remark 1. 1 If X is a Banach space, then as a metric space with $d(x, y) = \| x - y \|$, the mapping $W: X \times X \times I \to X$ defined by $W(x, y; \lambda) = \lambda x + (1 - \lambda)y$ is a convex structure. More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x + (1 - \lambda)\, y, 0) \leqslant \lambda d(x, 0) + (1 - \lambda)\, d(y, 0)$, it is a convex metric space. There are many other examples, but we consider these as paradigmatic.

Definition 1. 2 Let X be a convex metric space. A nonempty subset

$K \subseteq X$ is *convex* if $W(x, y, \lambda) \in K$, whenever $x, y \in K$ and $\lambda \in I$.

Remark 1. 2 Takahashi [8] has shown that the open spheres, $B(x, r) = \{y \in X: d(x, y) < r\}$, and closed spheres, $B[x, r] = \{y \in X : d(x, y) \leqslant r\}$ are convex Also, if $\{K_\alpha : \alpha \in \triangle\}$ is a family of convex subsets of X, then $\bigcap\{K_\alpha: \alpha \in \triangle\}$ is convex.

Definition 1. 3 A convex metrie space X is said to have property (C) if every decreasing net of nonempty, closed convex subsets of X has nonempty intersection.

Remark 1. 3 It is a theorem of Smulian [2, p. 433] that a convex subset of a Banach space has property (C) if and only if it is weakly compact.

Definition 1. 4 Let B be a bounded set in a convex metric space X, and let $\delta(B)$ be its diameter, *i. e.*, $\delta(B) = \sup \{d(x, y): x, y \in B\}$. A convex subset S of X is said to have normal structure if every bounded convex subset S_1 of S which contains more than one point has a point which is a nondiametral point of S_1, *i. e.*, there is a point x of S_1 such that $\sup \{ d(x, y) : y \in S_1\} < \delta(S_1)$.

It is easily seen that the following definition is equivalent to the above definition. A bounded, closed convex subset S of a metric space X is said to have normal structure if for each closed convex subset S_1 of S which contains more than one point, there exists an $x \in S_1$ and $\alpha(S)$, $0 < \alpha(S) < 1$, such that $\sup \{d(x, y) : y \in S_1\} = r_x(S_1) \leqslant \alpha(S_1)\, \delta(S_1)$.

2. Main Results

Theorem 2. 1 K be a nonempty, closed, bounded, convex subset of X. Assume K has normal structure. Let $T: K \to A(K)$ be a mapping

satisfying for each closed convex subset F of K invariant under T and with $\delta(F) > 0$, there exists some $\alpha(F)$, $0 < \alpha(F) < 1$, such that $\delta(Tx, Ty) \leqslant \max \{ \delta(x, F), \alpha(F)\, \delta(F) \}$ for all $x, y \in F$. Then T has a fixed point.

Proof Let G be the family of all nonempty, closed convex subsets of K, each of which is mapped into itself by T. Partially order G by set inclusion. Using property (C) and Zorn's lemma, we infer that G has a minimal element, say M. We show that M consists of a single point. Suppose M contains more than one element. By the definition of normal structure, there exists an $x_0 \in M$ and $\alpha_1(M)$, $0 < \alpha_1(M) < 1$, such that $\delta(x_0, M) = \sup \{ d(x_0, y) \colon y \in M \}$ $\leqslant \alpha_1(M)\, \delta(M)$. If $\delta(Tx, Ty) \leqslant \delta(x, M)$ for all $x, y \in M$, let $M_0 = \{x \in M \colon \delta(x, M) \leqslant \alpha_1\, \delta(M)\}$. Otherwise, by hypothesis, there exists $\alpha(M) > 0$ $0 < \alpha(M) < 1$, such that $\delta(Tx, Ty) \leqslant \alpha\delta(M)$ for some $x, y \in M$. Let $\beta = max\{\alpha, \alpha_1\}$ and $H = \{x \in M \colon \delta(x, M) \leqslant \beta\delta(M)\}$. H is nonempty, since $x_0 \in H$. Clearly, H is convex. Since $x \to \delta(x, M)$ is continuous, H is closed. Let $x \in H$, $\delta(Tx, Ty) \leqslant max\,\{\delta(x, M), \alpha\delta(M)\}$ $\leqslant \beta\delta(M)$ for $y \in M$. Hence, $T(M)$ is contained in a closed ball of center Tx and of radius $\beta\delta(M)$. By the minimality of M if $m \in Tx$, then $M \subset \bar{S}\,[m, \beta\delta(M)]$ (the closed spherical ball of center m and radius $\beta\delta(M)$), whence $m \in H$ and $T(H) \subset H$.

But $\delta(H) \leqslant \beta\delta(M) < \delta(M)$, which contradicts the minimality of M. Thus, M is a singleton, and this completes the proof.

Corollary 2. 1 Let K be a bounded, closed convex subset of X. Assume K has normal structure. Let T be a mapping of K into itself which satisfies for each closed convex subset F of K invariant under T, with $\delta(F) > 0$, there exists some $\alpha(F)$, $0 < \alpha(F) < 1$, such that

$d(Tx, Ty) \leqslant max \{ \delta(x, F), \alpha(F)\, \delta(F)\}$ for each $x, y \in X$, then T has a fixed point.

Corollary 2. 2 Let K be a nonempty, closed, bounded convex subset of X. Assume K has normal structure. Let T be a self-mapping of K satisfying: for each closed convex subset F of K invariant under T, with $\delta(F) > 0$, there exists some $\alpha(F)$, $0 < \alpha(F) < 1$, such that $d(Tx, Ty) \leqslant max \{ d(x, y), r(F)\ \ \alpha(F)\, \delta(F) \}$ for $x, y \in F$, then T has a fixed point.

The following example shows that our results generalize those of Browder [1], Kirk [5], and Takahashi [8].

Example 2. 1 Let $X = R$ with the usual norm and $K = [0, 4]$. Define $T : K \rightarrow K$ by

$$T(x) = \begin{cases} 0 \text{ for } x \neq 4 \\ 3 \text{ for } x = 4. \end{cases}$$

Each closed convex subset F of K invariant under T with $\delta(F) > 0$ is of the form $[0, r]$, $r \leqslant 4$. For $r < 4$, $T([0, r]) = \{0\}$, and in this case $| Tx - Ty | \leqslant max \{ | x - y |, r(F), \alpha\, \delta(F) \}$, for $x, y \in F$, is satisfied for any $\alpha > 0$. For $r = 4$, $T([0, r]) = \{0, 3\}$, hence for $\alpha = 4/5$, $| Tx - Ty | \leqslant max \{ | x - y |, r(K), \alpha\delta(K)\}$ for all $x, y \in K$ is satisfied with 0 as a fixed point of T. Since $| T4 - T3 | = | 3 - 0 | = 3$, $| T4 - T3 | \leqslant | 4 - 3 | = 1$ is not satisfied for $3, 4 \in K$. In fact, $| Tx - Ty | \leqslant max \{ | x - y |, r(F)\}$ is also not satisfied for $F = K$.

Theorem 2. 2 Let K be a closed, bounded, convex subset of X. Let $T_1, T_2 : K \rightarrow K$ be two mappings satisfying :

(i) $d(T_1x, T_2y) \leqslant max \{[d(x, T_1x) + d(y, T_2y)]/2,$

$[d(x, T_2y) + d(y, T_1x)]/3,$

$[d(x, y) + d(x, T_1x) + d(y, T_2y)]/3)$ for all $x, y \in K$;

(ii) $T_1A \subset A$ if and only if $T_2A \subset A$ for each closed convex subs[et] A of K with $\delta(K) > 0$:

(iii) either $\sup_{z \in A} d(z, T_1z) \leqslant \delta(A)/2$, or $\sup_{z \in A} d(z, T_2z) \leqslant \delta(A)/2$

for each closed convex subset A of K invariant under T_1 and T_2 with $\delta(A) > 0$. Then, T_1 and T_2 have a unique common fixed point.

Proof Let G be the family of all nonempty, closed, convex subsets of K, each of which is mapped into itself by T_1 and T_2. Partially order G by set-inclusion. By property (C) and Zorn's Lemma, it follows that K has a minimal element, say F. Suppose $\delta(F) > 0$. Without loss of generality, we may assume that $\sup_{z \in F} d(z, T_2z) \leqslant \delta(F)/2$. Let $x \in F_c$. Since $x \in F_c$, $\delta(x, F) = r(F)$. Also, $\delta(F)/2 \leqslant r(F)$. Therefore, for $y \in F$, $\frac{1}{2}[d(x, T_1x) + d(y, T_2y)] \leqslant \frac{1}{2}[\delta(x, F) + \sup_{z \in F} d(z, T_2z)]$
$\leqslant \frac{1}{2}[r(F) + \delta(F)/2] \leqslant r(F)$. Likewise, $1/3\,[d(x, T_2y) + d(y, T_1x)]$
$\leqslant 1/3\,[r(F) + r(F) \leqslant \delta\, r(F)$, and $1/3\,[d(x, y) + d(x, T_1x) + d(y, T_2y)]$
$\leqslant 1/3\,[r(F) + r(F) + \delta(F)/2] \leqslant r(F)$. Hence, $d(T_1x, T_2y) \leqslant \max$
$\{[d(x, T_1x) + d(y, T_2y)]/2, [d(x, T_2y) + d(y, T_1x)]/3 + [d(x, y) + d(x, T_1x) + d(y, T_2y)]/3\} \leqslant r(F)$. So, $T_2(F) \subset \bar{B}[T_1x, r(F)]$ and, hence, $T_2(F \cap \bar{B}) \subseteq F \cap \bar{B}$ as $T_2(F) \subseteq F$. By hypothesis (ii), it follows that $T_1(F \cap \bar{B}) \subset F \cap B$. By the minimality of F, we conclude that $F \subseteq \bar{B}$. This gives $\delta(T_1x, F) = r(F)$, whence, $T_1x \in F_c$. Therfore, $T_1(F_c) \subset F_c$, and by hypothesis (ii), $T_2(F_c) \subset F_c$. We claim that F_c is a proper subset of F. Suppose not, that is, $F = F_c$. Since $\delta(x, F) = r(F)$ for each $x \in F$, we obtain $\delta(F) = r(F) = \delta(x, F)$. From (i), we get $d(T_1x, T_2y) \leqslant \max\{[d(x, T_1x) + d(y, T_2y)]/2, [d(x, T_2y) + d(y, T_1x)]/3, [d(x, y) + d(x, T_1x) + d(y, T_2y)]/3\} < \max\{3\,\delta(F)/4, [\delta(F) + \delta(F)]/3, [\delta(F) + \delta(F) + \delta(F)/2]/3\}$
$= 5\,\delta(F)/6$. Similarly, $\delta(T_1x, F) \leqslant 5\,\delta(F)/6 < \delta(F)$,

which is a contradiction. Consequently, if F contains more than one element, then F_c is a proper subset of F, a contradiction to the minimality of F. Hence, F contains exactly one element, say, x_0, whence $T_1x_0 = x_0 = T_2x_0$. Unicity of the fixed point is clear.

Theorem 2.3 Let K be a nonempty, closed, bounded, convex subset of X. Suppose K has normal structure. Let $T_1, T_2 : K \rightarrow K$ be two mappings satisfying :

(i) for each closed convex subset F of K invariant under T_1 and T_2, with $\delta(F) > 0$, there exists some $\alpha_1(F)$, $0 < \alpha_1(F) < 1$, such that $d(T_1x, T_2y) \leqslant max \{[d(x, T_1x) + d(y, T_2y)]/2, [d(x, T_2y) + d(y, T_1x)]/3, [d(x, y) + d(x, T_1x) + d(y, T_2y)]/3, \delta(x, F), \alpha_1(F)\, \delta(F)\}$ for $x, y \in F$.

(ii) $T_1A \subset A$ if and only if $T_2A \subset A$ for each closed convex subset A of K with $\delta(A) > 0$,

(iii) for each closed convex subset B of K invariant under T_1, T_2 with $\delta(B) > 0$, there exists some $\alpha_2(B)$, $\frac{1}{2} \leqslant \alpha_2(B) < 1$, such that either $\sup_{z \in B} d(z, T_1z) \leqslant max \{ r(B), \alpha_2(B)\, \delta(B) \}$. Then, T_1 and T_2 have a common fixed point

Proof As in the proof of Theorem 2.2, there exists a minimal element F. Suppose F contains more than one element. By the definition of normal structure, there exists $x_0 \in F$ such that

$$sup \{d(x_0, y) : y \in F\} = \delta(x_0, F) = r_{x_0}(F) \leqslant \alpha_3 (F)\, \delta(F)$$

for some α_3, $0 < \alpha_3 < 1$. Without loss of generality, we assume that $\sup_{z \in F} d(z, T_2z) \leqslant \max \{ r(F), \alpha_2\, \delta(F)\}$ for some α_2, $\frac{1}{2} \leqslant \alpha_2 < 1$. If

$d(T_1x, T_2y) \leqslant \max \{ [d(x, T_1x) + d(y, T_2y)]/2,$
$[d(x, T_2y) + d(y, T_1x)]/3, [d(x, y) + d(x, T_1x) + d(y, T_2y)]/3, r_x(F)\}$

for all $x, y \in F$, let $\beta = \max \{\alpha_2, \alpha_3\}$ and $F_0 = \{x \in F: r_x(F) \leqslant \beta\delta(F)\}$. Otherwise, by hypothesis (i), there exists $\alpha_1(F)$,

$0 < \alpha_1(F) < 1$, such that $d(T_1x, T_2y) \leqslant \alpha_1(F)\, \delta(F)$ for some $x, y \in F$. Let $\beta = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and $F_0 = \{x \in F: r_x(F) \leqslant \beta\delta(F)\}$. F_0 is non-empty, since $x_0 \in F_0$. Clearly, F_0 is convex and closed. Let $x \in F_0$. Then, $d(T_1x, T_2y) \leqslant \max\{[d(x, T_1x) + d(y, T_2y)]/2,$
$[d(x, T_2y) + d(y, T_1x)]/3, [d(x, y) + d(x, T_1x) + d(y, T_2y)]/3, r_x(F),$
$\alpha_1(F)\, \delta(F)\} \leqslant \beta\delta(F)$ for $y \in F$. The same argument as in Theorem 2. 2 yields $F_0 \in G$. But,
$\delta(F_0) \leqslant \beta\delta(F) < \delta(F)$, which contradicts the minimality of F. Hence, F contains exactly one point, which is a common fixed point of T_1 and T_2.

Remark 2. 1 The following example shows that normal structure alone is not sufficient to establish the existenee of a common fixed point for T_1 and T_2. In other words, the condition (iii) cannot be dispensed from Theorem 2 3.

Example 2. 2 Let $X = R$ with the usual metric, and Let $K = [0, 4]$. Define $T: K \rightarrow K$ as follows : $To = 4$ and $Tx = 0$ for . $x \neq 0$. Clearly, K, being a compact convex subset, has normal structure. Moreover, K is weakly compact and, hence, has property (C) . Taking $T_1 = T_2 = T$, it follows easily that T satisfies all conditions, except (iii) of Theorem 2. 3, and is without fixed point .

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A THEOREM ON THE EXISTENCE OF POSITIVE EIGENVALUES OF k-SET CONTRACTIONS (*)

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In this note we prove a result on the existence of non-zero eigenvalues for k-set contractions which satisfy a Birkhoff-Kellog type condition on a bounded open subset of a special wedge (see definition below).

If we restrict our maps acting in cones, then analogous results obtained previuosly by several authors (cfr. [3, 5, 6]).

To be more specific, let X be a real Banach space, P a special wedge in X, O a bounded open subset of X containing $0 \in X$, $\Omega = O \cap P$ and $f: \Omega \to P$ a k-set contraction with $k < 1$. Using some elementary properties of the topological degree and a result of Fournier-Peitgen [2], we succeeded in showing the following result:
Let $f: \overline{\Omega} \to P$ be a k-set contraction satisfying the condition

$$inf \{ \| f(x) \| : x \in \partial\Omega\} > k \; sup \{ \| x \| : x \in \partial\Omega\},$$

then there exist an eigenvalue $\lambda > 0$ and an eigenvector $x \in \partial\Omega$ with $f(x) = \lambda x$.

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In order to prove the above result we recall, for sake of completeness, some basic definitions.

Let (M, d) be a metric space. By the Kuratowski measure of non compactness of set $A \subset M$ is meant the $\alpha_M(A)$–number

$$\alpha_M(A) = \inf\{\, \varepsilon > 0 : A = \bigcup_{i=1}^{n} S_i \,,\ n < \infty \text{ and diameter } (S_i) \leqslant \varepsilon$$

for $1 \leqslant i \leqslant n\}$.

Let $f\colon M \to N$ be a continuous map where (M, d) and (N, d') are metric spaces. By the measure of non compactness of the map f (cfr. [1]) is meant the $\alpha(f)$–number:

$$\alpha(f) = \inf\{k : \alpha_M(f(A)) \leqslant k\ \alpha_M(A) \text{ for all } A \subset M\}.$$

Let Ω be a subset of M. A continuous map $f\colon \Omega \to M$, is called a strict-set contraction if $\alpha(f) < 1$. We will say that the map f is a k–set contraction if $k \geqslant \alpha(f)$.

Let X be a real linear normed space. Following Schaefer [7], a nonempty closed, convex subset P of X is called a *wedge* if $x \in P$ implies $tx \in P$ for $t \geqslant 0$.

For $r > 0$, we set

$$B_r = \{\, x \in P : \| x \| \leqslant r\}$$

and

$$\partial B_r = \{\, x \in P : \| x \| = r\}.$$

We recall that a *linear ordering* of X associated with a wedge P is the relation $\leqslant$ defined by

$$x \leqslant y \quad \textit{implies} \quad y - x \in P.$$

By an *ordered linear space X* we shall mean a real linear space with a linear ordering.

Let X be an infinite dimensional real Banach space with norm $\|\cdot\|$. We call a *special wedge* (see [2]), if P is a wedge for which there exists $y_0 \in P$, $y_0 \neq 0$, such that

$$\| x + y_0 \| \geqslant \| x \|$$

for all $x \in P$. This is a crucial assumption on deriving our main result.

In fact, in a special wedge P, the following lemma, proved by Fournier-Peitgen in [2, Lemma 1. 5. 1, p. 173], holds:

Lemma 0- *Let P be a special wedge in a linear normed space X and let $r > 0$. For all $\varepsilon > 0$, there exists a retraction*

$\rho_\varepsilon : B_r \to \partial B_r$ *such that*

$$\rho_\varepsilon /_{\partial B_r} = Id_{\partial B_r}$$

and $\qquad \alpha(\rho_\varepsilon) \leqslant 1 + \varepsilon.$

Let Ω be a bounded, relatively open subset of a special wedge P (*i. e.*, $\Omega = O \cap P$ for some open subset θ of X). By $\overline{\Omega}$ and $\partial\Omega$ we denote the clousure and the boundary of Ω in P, respectively.

If $f : \overline{\Omega} \to P$ is a k-set contraction, we denote with $i\,(f, \Omega)$ the fixed point index of f in Ω (see [4]).

The first result that we shall prove is the following :

Theorem 1. *Let (X, P) be an infinite dimensional ordered real Banach space, P a special wedge and let Ω be a bounded, relatively open subset of P containing 0. Let $f: \overline{\Omega} \to P$ be a strict set-contraction.*

Assume that

$i)\ \delta = inf\,\{\, \| f(x) \| : x \in \partial\Omega \} > sup\,\{\, \| x \| : x \in \partial\Omega \} = d.$

Then

$$i(f, \Omega) = 0.$$

Poof. Let us note first of all that if condition $i)$ of Theorem 1 is replaced by the stronger one:

$$i') \qquad \delta' = inf \,\| \{ (x) \| : x \in \overline{\Omega} \} > sup\,\{\, \| x \| : x \in \partial\Omega \} = d,$$

then the conclusion of the theorem is straightforward, *i. e.*,

$$i(f, \Omega) = 0.$$

Indeed, if we assume the contrary, by the solution property of the index, there exists $x_0 \in \overline{\Omega}$ such that $f(x_0) = x_0$. But then

$$\delta' \leqslant \| f(x_0) \| \leqslant \| x_0 \| \leqslant d,$$

contradicting $i')$.

Clearly, $i)$ can be replaced by the following inequality :

$$i'')\ \delta > (1 + \varepsilon_0)d, \text{ for some } \varepsilon_0 > 0.$$

Being $\alpha(f) < 1$, there exists some positive $\varepsilon < \varepsilon_0$ such that $(1 + \varepsilon)\,\alpha(f) < 1$.

Now, by Lemma 0, we can choose a retraction ρ_ε of B_δ onto its boundary with $\alpha(\rho_\varepsilon) \leqslant 1 + \varepsilon$. Define a map $\rho : P \to P$ by

$$\rho(x) = \begin{cases} \rho_\varepsilon(x) & if \quad x \in B_\delta \\ x & if \quad x \in P/B_\delta. \end{cases}$$

Clearly ρ is a $(1 + \varepsilon)$-set contraction.

Moreover, the map $g: \overline{\Omega} \to P$ defined by

$$g(x) = \rho(f(x)) \quad \text{for} \quad x \in \overline{\Omega}$$

has the following properties:

$$\| g(x) \| \geqslant \delta \quad \text{for any} \quad x \in \overline{\Omega}$$

and

$$\alpha(g) \leqslant \alpha(\rho)\, \alpha(f) \leqslant (1 + \varepsilon)\, \alpha(f).$$

Hence, by our choice of ε, we obtain that $\alpha(g) < 1$, *i.e.*, g is a strict-set contraction satisfying i').

Thus

$$i(g, \Omega) = 0.$$

But the restriction of g to $\partial\Omega$ coincides with the map f. therefore by the invariance property of the fixed point index (upon the boundary) we obtain that

$$i(f, \Omega) = 0.$$

Remark. The same result holds in an infinite dimensional real Banach space X for which for all $\varepsilon > 0$ there exists a retraction ρ_ε of the ball

$$B = \{x \in X: \| x \| \leqslant 1\}$$

onto its boundary

$$\partial B = \{x \in X: \| x \| = 1\}$$

such that

$$\alpha(\rho_\varepsilon) \leqslant 1 + \epsilon.$$

The characterization of spaces with this properties will be the object of a forthcoming paper.

For a k-set contraction, $k < 1$, we derive the following result:

Corollary. *Let (X, P) be an infinite dimensional ordered real Banach space, P a special wedge and let Ω a bounded, relatively open subset of P containing 0. Let $f: \overline{\Omega} \to P$ be a k-set contraction, $0 \leqslant k < 1$, such that*

i_0) $$\delta = inf \{ \| f(x) \| : x \in \partial\Omega \} > k \sup \{ \| x \| : x \in \partial\Omega \} = kd;$$

ii_0) $$\lambda x \neq f(x) \text{ for } k \leqslant \lambda \leqslant 1 \text{ and } x \in \partial\Omega$$

then

$$i(f, \Omega) = 0.$$

Proof.

Let $\varepsilon > 0$ such that $\delta > (k + \epsilon) d$ and $k + \epsilon < 1$.

Consider the map $\frac{1}{k + \epsilon} f : \overline{\Omega} \to P$.

Clearly, this map is a strict-set contraction which satisfies assumption i) of Theorem 1 and so we have that

$$i\left(\frac{1}{k+\varepsilon} f, \Omega\right) = 0.$$

Consider the homotopy $H : \overline{\Omega} \times \left[1, \frac{1}{k+\varepsilon}\right] \to P$ defined by

$$H(x, t) = x - tf(x).$$

By ii_0) H is an admissible homotopy in the sense of [4] hence we have that

$$i(f, \Omega) = i\left(\frac{1}{k+\varepsilon} f, \Omega\right) = 0.$$

This completes the proof.

We show now how, using the previous results, we can obtain the existence of positive eigenvalues for k-set contraction in special wedgs with satisfying the Birkhoff-Kellog type condition (cfr. [3, 5, 6]).

Theorem 2. *Let (X, P) be an infinite dimensional ordered real Banach space, P a special wedge, Ω a bounded relatively open subset of P containing* 0. *If $f\colon \overline{\Omega} \to P$ is a k-set contraction for which*

1) $\inf \{ \| f(x) \| : x \in \partial\Omega\} > k \sup \{ \| x \| : x \in \partial\Omega\}$,

then there exist $\lambda > 0$ and $x \in \partial\Omega$ such that

$$\lambda x = f(x).$$

Proof. Assume the contrary, *i. e.*, for all $\lambda > 0$ and for all $x \in \partial\Omega$,

$$\lambda x \neq f(x).$$

Now, let $\varepsilon > 0$ such that $\delta > (1 + \varepsilon)d$. Consider the map

$$\frac{1}{k + \varepsilon} f : \overline{\Omega} \to P.$$

Clearly, this map is a strict-set contraction satisfying assumption $i)$ of Theorem 1. Hence

$$i\left(\frac{1}{k + \varepsilon} f, \Omega\right) = 0.$$

On the other hand, for all

$$0 \leqslant t \leqslant \frac{1}{k + \varepsilon},$$

the homotopy $H : \overline{\Omega} \times \left[0, \frac{1}{k+\varepsilon}\right] \to P$

defined by $H(x, t) = H_t(x) = x - tf(x)$

for $x \in \overline{\Omega}$ and for for $t \in \left[t, \frac{1}{k+\varepsilon}\right]$

is admissible since $\lambda x \neq f(x)$ for all $\lambda \geqslant k + \varepsilon$.

Hence $i\left(\frac{1}{k+\varepsilon} f, \Omega\right) = i(H_0, \Omega) = 1$,

which is a contradiction.

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A DEGREE THEORY FOR WEAKLY CONTINUOUS MULTIVALUED MAPS IN REFLEXIVE BANACH SPACES*

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ABSTTRACT

In this paper we propose a degree theory for multivalued weakly upper semicontinuous maps with convex values in a reflexive Banach space with Schauder basis.

INTRODUCTION

A degree theory for singlevalued weakly continuous maps was developed in [4] The aim of this note is to introduce a degree theory for multivalued weakly upper semicontinuous maps (see Definition 1. 5). Our results extend the degree theory of singlevalued weakly continuous maps of [4].

The paper is divided into three parts.

In the first part we give the notations and the basic definitions to be used in the sequel and we introduce the so-called condition (A) which plays the same role in our theory as the usual condition "the

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map has no fixed points on the boundary" in the case of maps upper semicontinuous with convex compact values.

The second part is devoted to the degree theory for the multivalued weakly upper semicontinuous maps with convex values, satisfying the condition (A) with respect to some Schauder basis in reflexive Banach spaces.

Finally, in the third part we introduce a class of multivalued maps which satisfies the condition (A) with respect to every orthonormal basis of a Hilbert space.

1. NOTATIONS, DEFINITIONS AND PRELIMINARY RESULTS

We recall that a multivalued map of a set X into a set Y is a triple (G, X, Y), where G, the graph of T, is a subset of $X \times Y$ such that $Tx = \{y \in Y; (x, y) \in G\}$ is nonempty for each $x \in X$. $TX = \{Tx: x \in X\}$ is the range of T, while X is its domain. We shall use the symbol $T : X \rightarrow Y$ to indicate a multivalued map. If $D \subset X$ then $TD = \{Tx: x \in D\}$.

Let X and Y be topological spaces and $T : X \rightarrow Y$.

Definition 1. 1.

The map T is called upper semicontinuous (u. s c.) at $x_0 \in X$ if for any open set V containing Tx_0 there exists a neighbourhood U of x_0 such that $x \in U$ implies $Tx \subset V$.

Definition 1. 2.

The map T is called u. s. c. on X if it is u. s. c. at each point $x \in X$ and Tx is compact for every $x \in X$.

Definition 1. 3.

The map T is called closed if its graph is closed in $X{\times}Y$.

Definition 1. 4.

The map T is called closed if its graph is closed in $X \times Y$.

A fixed point of a multivalued map $T : X \to Y$ is a point $x \in X$ such that $x \in Tx$.

Let X be a real reflexive Banach space with norm $\| \cdot \|$, which admits a Schauder basis. By B_r we denote the ball $\{x \in X : \| x \| \leqslant r\}$, where r is a real positive number, with ∂B_r the sphere $\{x \in X : \| x \| = r\}$. We shall also use the symbols "—o" and "$\rightharpoonup$" to denote the strong and the weak convergence in X respectively. Moreover, we shall denote by τ the weak topology on X.

Definition 1. 5.

The map $T : B_r \to X$ is called weakly upper semicontinuons (*w. u. s. c.*) *at $x_0 \in S_r$, if it is u. s. c. at x_0, with respect to the weak topology τ.*

Definition 1. 6.

The map $T : S_r \to X$ is called w. u. s. c. on B_r, if it is w. u. s. c. at each point $x \in B_r$ and Tx is weakly compact (i. e. compact with respect to τ) *for every $x \in S_r$.*

Definition 1. 7.

The map $T : B_r \to X$ i s called weakly closed, if its graph is closed in $B_r \times X$ with respect to the weak topology on the product.

Remark 1. 1.

If $T : S_r \to X$ is weakly closed then (see [1]) :

$$\left.\begin{array}{l} \{x_m\} \subset S_r, \ x_n \rightharpoonup x \\ \qquad\qquad\quad y_n \rightharpoonup y \\ \forall \ n \in N, \ y_n \in Tx_n \end{array}\right\} \Rightarrow y \in Tx.$$

Definition 1. 8.

Let $D \subset X$. The map $T: D \to X$ is called bounded on D, if it maps bounded sets of D into bounded sets of X.

Remark 1. 2.

Let $D \subset X$. If D is weakly compact then D is bounded and weakly closed (*i. e.* closed with respect to τ), by the reflexivity of X.

Remark 1. 3.

If $T : S_r \to X$ is *w. u. c.* on S_r then it is weakly closed (see [1]).

Remark 1. 4.

If $T : X \to X$ is *w. u. s. c.* then it sends weakly compact sets of X into weakly compact sets of X (see [1]).

Remark 1. 5.

If $T : B_r \to X$ is *w. u. s. c.* on S_r then Tx is closed for any $x \in B_r$.

Proposition 1. 1.

Let $D \subset X$. If $T : D \to X$ is w. u. s. c. and D is weakly closed then T is bounded on D.

Proof : Given a bounded subset $K \subset D$, its weak closure $\bar{\bar{K}}$ (*i. e.* the closure of K with respect to τ) is weakly compact, by the reflexivity of X. Moreover, $\bar{\bar{K}}$ is contained in D, since D is weakly closed. Consequently, $T\bar{\bar{K}}$ is weakly compact (see Remark 1. 4). Form the Remark 1. 2 it follows that $T\bar{\bar{K}}$ is bounded and weakly closed. Hence $TK \subset T\bar{\bar{K}}$ is bounded "

Let X be a real reflexive Banach space, $\{e_i\}$ a Schauder basis of X and $\{\phi_i\}$ the corresponding biorthogonal system in the dual space X^*, *i. e.* $\phi_i(e_j) = \delta_{ij}$. Given $n \in N$, by $\partial_m S_r$ we denote the

n-dimensional sphere $\{ x \in \partial S_r : \sum_{i=1}^{n} | \phi_i(x) |^2 = r^2 \}$.

We say that the map $T : B \to X$ satisfies the condition (A), with respect to the Schauder basis $\{e_i\}$, if there exists $n_0 \in N$ such that, for all $n \geqslant n_0$ and for all $x \in \partial_n S_r$ and $y \in Tx$, there exists $j \leqslant n$ such that $\phi_j(y) \neq \phi_j(x)$.

In particular, we say that the map $T : B \to X$ satisfies the condition (B), if it satisfies the condition (A) with $n_0 = 1$.

Finally, the map $T : B \to X$ satisfies the condition (C), if $x \notin Tx$ for all $x \in \partial S_r$.

The following examples prove the mutual independence of the condition (A) and (C).

Example 1. 1. ((A) does not imply (C)).

Let H be a real and separable Hilbert space with inner product $(,)$ and orthonormal basis $\{e_i\}$. Fix a $\in H$ such that $\| a \| = 2r - \varepsilon$, where $\varepsilon < r$ is a real and positive number. We define $Tx = S_\varepsilon + a - x$, where $S_\varepsilon = \{ z \in H : \| z \| \leqslant \varepsilon \}$, for all $x \in S_r$. The map T has a fixed point on ∂S_r. Let $x = [r/(2r - \varepsilon)]a$ and $z = [\varepsilon/(2r - \varepsilon)]a$, then we have $\bar{x} = z + a - \bar{x}$ with $\| x \| = r$ and $\| z \| = \varepsilon$. Consequently, $\bar{x}$ is a fixed point of T on ∂S_r and thus (C) does not hold. On the other hand the map T verifies the condition (A) with respect to every orthonormal basis $\{e_i\}$ with the property that $(a, e_j) \neq 0$ for infinitely many indices j. Suppose now that T does not satisfy the condition (A). Then one can find x and z such that $(z + a - x) \in Tx$, $x \in \partial_n S_r$, and $(z + a - x, e_j) = (x, e_j)$, for $j = 1,\ldots,n$. By the second and, third property, $\sum_{j=1}^{n} [(z, e_j) + (a, e_j)]^2 = 4r^2$; on the other hand

$$\sum_{j=1}^{n} [(z, e_j) + (a, e_j)]^2 < 4r^2, \text{ since } (a, e_j) \neq 0 \text{ for infinitely many } j.$$

This contradiction completes the **proof.**

Example 1. 2. ((C) does not imply (A))

Let $\{e_i\}$ be an orthonormal basis of H and $\varepsilon < r$ a real positive number. We define, for all $x \in S_r$

$$Lx = \{h \in H : h = x + \sum_{i=1}^{\infty} (y, e_i)\, e_{i+1} \text{ for some } y \in S_\varepsilon + x\}.$$

It is easy to verify that the map L is without fixed points on ∂S_r. The map L, however, does not satisfy the condition (A) with respect to the orthonormal basis $\{e_i\}$. In fact, for all $n \in N$, the element $re_n \in \partial_n S_r$ is such that

$$(re_n, e_j) = (re_n + \sum_{i=1}^{\infty} (re_n, e_i)\, e_{i+1}, e_j) \text{ for } j = 1, \quad ,n.$$

Remark 1. 6.

Obviously, in the definition of condition (A), the choice of the index n_0 depends, in general, on the basis. Moreover the example 1. 1 shows that n_0 may tend to $+\infty$, if the orthonormal basis $\{e_i\}$ changes.

Proposition 1 2.

If $T : B_r \to X$ is w. u. s. c. and does not satisfy the condition (A) with respect to a Schauder basis $\{e_i\}$, then T has fixed point in S_r.

Proof : If T does not satisfy the condition (A) there exists an infinite sequence $\{n_k\}$ of positive intergers with $n_k \to +\infty$ and a sequence $\{x^k\}$ with $x^k \in \partial_{n_k} B_r$,

for all $k \varepsilon N$, such that $\phi_j(y) = \phi_j(x^k)$ for $j=1, \ldots, n_k$ for some $y \varepsilon Tx_k$. By contruction, there exists $ze^{n_k} \in Tx^{n_k}$,

such that $z^{n_k} = x^{n_k} + y^{n_k}$ with $x^{n_k} = \sum_{i=1}^{n_k} \phi_i (x^{n_k}) e_i$

and $y^{n_k} = \sum_{i=n_k+1}^{\infty} \phi_i (z^{n_k}) e_i$, since $\{x^{n_k}\}$ is bounded there

exists a subsequence $\{x^{n_{k(s)}}\}$ of $\{x^{n_k}\}$ such that $x^{n_{k(s)}} \to x \in S_r$. Moreover, the corresponding subsequence $\{z^{n_{k(s)}}\}$ of $\{z^{n_k}\}$ converges also weakly to x. In fact, this follows from the bounded ness of the sequence $\{z^{n_{k(s)}}\}$and the map T, (see Proposition 1.1), since $(z^{n_{k(s)}}, e_j) - o\ (x, e_j)$ for all $j \in N$. Finally, since the graph of T is weakly closed, one has $x \in Tx$ (see Remark 1 1)

2. Definition of a Topological Degree for Multivalued W. U. S. C. Maps

Let $T : B_r \to X$ be w. u. s. c., and $\{e_i\}$ a Schauder basis in X with biorthogonal system $\{\phi_i\}$ (see before). Let $X_n = [e_1, \ , e_n]$ the linear hull of the first n basis vectors, B_r^n the sphere of radius r in X_n, and ∂B_r^n its boundary For fixed $n \in N$, we define $T_n : B_r^n \to X_n$ by

$$T_n (x_1, \ldots, x_n) = \{ (y_1, \ , y_n) : y \in Tx\},$$

where $x_i = \phi_i(x)$ and $y_i = \phi_i(y)$ for $i = 1, \ldots, n$.

If $T : S_r \to X$ is w. u. s. c. and satisfies the condition (A) with respect to the Schauder basis $\{e_i\}$, then $T_n : B_r^n \to X_n$ is fixed point free on

∂B_r^n, for all $n \geqslant n_0$. In fact, if we had $x \in T_n x$ for some $x \in \partial B_r^n$ there would exist $y \in Tx$ with $x = \sum_{i=1}^{n} \phi_i(x)\, e_i$ and hence $\phi_j(y) = \phi_j(x)$ for $j = 1, .., n$, contradicting the condition (A). Moreover, T_n is u. s. c. for all $n \in N$, as composition of map T with imbeddings and projections.

Definition 2. 1.

Let $T : B_r \to X$ be w. u s. c, with convex values. If T satisfies the condition (A), with respect to the Schauder basis $\{e_i\}$, we define Deg(I–T, S_r, 0), the degree of T on S_r with respect to 0, as follows: Let $\hat{Z}$ be the set of all integers togethers with $\{+\infty\}$ and $\{-\infty\}$. Then Deg (I-T, S_r, 0) is defined to be the subset of $\hat{Z}$ given by:

$$\text{Deg}\,(I\text{-}T, S_r, 0) = \{\gamma \in \hat{Z} / \text{there exists an infinte sequence } \{n_k\}$$

of positive integers with $n_k \to \infty$ such that $\text{Deg}\,(I\text{-}T_{n_k}, B_r^{n_k}, 0) \to \gamma\}$.

Remark 2. 1.

Deg (I-T, S_r, 0) $\neq \phi$, since $\hat{Z}$ is compact. In particular, if γ is a (finite) integer, then Deg (I–T_{n_k}, $B_r^{n_k}$, 0) $\to \gamma$ iff Deg (I–T_{n_k}, $B_r^{n_k}$, 0) $= \gamma$ for all but a finite number of n_k.

Remark 2. 2.

The degree Deg (I–T_{n_k}, $B_r^{n_k}$, 0) used in Definition 2. 1 is the Cellina –Lasota degree [3]. It is well defined for all $n_k \geqslant n_0$.

In fact, the map T is w. u. s. c., bounded, with convex and closed

values (see Proposition 1. 1 and Remark 1. 5). Hence

$I\text{-}T_{n_k} : B_r^{n_k} \rightarrow X_{n_k}$ is u s. c., with convex and compact values.

The following theorems gather the main properties of the degree of Definition 2. 1.

Theorem 2. 1. (Solution property).

Let $T : B_r \rightarrow X$ be w. u. s. c., with convex values, and such that condition (A) holds with respect to the Schauder basis $\{e_i\}$. If Deg $(I\text{-}T, B_r, 0) \neq \{0\}$, then there exists an element $x \in B_r$ such that $x \in Tx$.

Proof : If Deg $(I\text{-}T, B_r, 0) \neq \{0\}$ there exists an infinite sequence $\{n_k\}$ of positive integers with n_k —o $+ \infty$ such that deg $(I\text{-}T_{n_k}, B_r^{n_k}, 0) \neq 0$. By the solution property of the Cellina-Lasota degree it follows that for every n_k there exists $x^{n_k} \in B_r^{n_k}$ such that $x^{n_k} \in T_{n_k} x^{n_k}$. The remaining part of the proof is analogues to that of the Proposition 1. 2.

Theorem 2. 2. (Homotopy invariance).

Let $H : B_r \times [0 \ 1] \rightarrow X$ be a w. u. s. c. map such that $H(x, t)$ ts convex for every $(x, t) \in B_r \times [0, 1]$. If the family of the maps $H_t = H(\cdot, t)$: $B_r \rightarrow X$ verifies the condition (A) for all $t \in [0, 1]$, with respect to the Schauder basis $\{e_i\}$, and if there exists $\bar{n} \in N$ such that $n_0(H_t) \leqslant \bar{n}$, for all $t \in [0, 1]$, then Deg $(I\text{-}H_t, B_r, 0)$ is independent of $t \in [0, 1]$.

Proof : For all $n \geqslant \bar{n}$, H_n: $B_r^n \times [0, 1] \rightarrow X_n$ is a homotopy such that $0 \notin (\cdot, t))(\partial B_r^n)$ for $0 \leqslant t \leqslant 1$. By the homotopy property of the Cellina-Lasota degree, we have that Deg $(I\text{-}H_n(\cdot, t), B_r^n, 0)$ is independent of $t \in [0, 1]$. Hence also Deg $(I\text{-}H_t, B_r, 0)$ is independent of $t \in [0, 1]$.

Theorem 2. 3 (Excision property)

Let $T: B_r \to X$ be w. u. s. c., with convex. values. If T satisfies the condition (A) on ∂S_λ, with respect to the Schauder basis $\{e_i\}$, for any λ with $r \leqslant \lambda \leqslant R$, then $Deg\,(I-T, B_r, 0) = Deg\,(I-T, B_R, 0)$.

Proof : We have that, for all $n \geqslant n_0$, $\deg(I-T_n, B^n_r, 0) = \deg(I-T, B^n_R, 0)$

since $0 \notin (I-T_n)\,(B^n_R \setminus \overset{\circ}{B}{}^n_r)$, where $\overset{\circ}{B}{}^n_r = B^n_r \setminus \partial B^n_r$.

Theorem 2. 4. (Borsuk type theorem).

Let $T : B_r \to X$ be w. u. s. c., with convex values If T satisfies the condition (A), with respect to the Schauder basis $\{e_i\}$, and is odd on ∂B_r, then $Deg(I-T, B_r, 0)$ is odd (i. e., $2m \notin Deg(I-T, B_r, 0)$ for any integer m). In particular, $\{0\} \neq Deg(I-T, B_r, 0)$ so that the equation $x \in Tx$ has a solution in B_r.

Proof : For all $n \in N$, $T_n : B^n_r \to X_n$ is an $u.\ s.\ c.$ map, with compact and convex values, which is odd on ∂B^n_r. Hence, for all $n \geqslant n_0$,

$\deg(I-T_n, B^n_r, 0)$ is odd. In fact, the homotopy

$$H_n(x, t) = [1/(1+t)]\,(I-T_n)\,x + [t/(1+t)]\,(-I+T_n)\,x$$

is admissible in the sense of [3], hence

$$\deg\,(H_n(\cdot\,, 0)\ B^n_r, 0) = \deg\,(H_n(\cdot, 1),\ B^n_r, 0)$$

and there exists $\bar{n} \in N$ such that $n(\lambda) \leqslant \bar{n}$, for all $\lambda \in [r, R]$,

where $H_n(\cdot\,, 1)$ is an odd map on ∂B^n_r. The fact that deg $(H_n(\cdot, 1), B^n_r, 0)$ is an odd integer can be shown by constructing a sequence (see [5], Lemma 2.3) of singlevalued continuous odd maps $f^n_k : B^n_r \rightarrow \overline{\text{co}}\ R(H_n(\cdot\,, 1))$, where $\overline{\text{co}}\ R(H_n(\cdot\,, 1))$ is the closed convex hull of the range $R(H_n(\cdot\,, 1))$ of the map $H_n(\cdot\,, 1)$, converging to $H_n(\cdot\,, 1)$ in the sense of [3]. Consequently, Deg $(I-T, B_r, 0)$ contains only odd integers.

Theorem 2.5. (Boundary value dependence).

Let $T, L : B_r \rightarrow X$ *be two w. u. s. c. maps, with convex values, which satisfy the condition* (A), *with respect to the Schauder basis* $\{e_i\}$. *If* $Tx = Lx$ *for all* $X \in \partial B_r$, *then* Deg $(I-T, B_r, 0) =$ Deg $(I-L, B_r, 0)$.

Proof : For all $n \in N$ and for all $x \in \partial B^n_r$, $T_n\ x = L_n x$. For all $n \geqslant \bar{n}$ $= \max\ \{n_o(T), n_o(L)\}$, deg $(I-T_n, B^n_r, 0)$ and deg $(I-L_n, B^n_r, 0)$ are well defined. Hence the homotopy

$$H_n(x, t) = (1-t)(I-T_n)\,x + t\,(I-L_n)x$$

is admissible in the sense of [3] for all $n \geqslant \bar{n}$. Therefore

$$\deg(I-T_n, B^n_r, 0) = \deg(I-L_n, B^n_r, 0) \text{ for all } n \geqslant \bar{n},$$

and hence Deg $(I-T, B_r, 0) =$ Deg $(I-L, B_r, 0)$.

3. A SPECIAL CLASS OF MULTIVALUED MAPS

In this section, let H denote a real separable Hilbert space. We give a sufficient condition for a map T to satisfy condition (A) with respect to every orthonormal basis $\{e_i\}$ in H. To this and, we first prove following

Proposition 3. 1.

For a map $T : B_r \rightarrow H$ the following assertions are equivalent

(i) *The map T satisfies the condition (B), with respect to every orthonormal basis $\{e_i\}$.*

(ii) *For all $x \in \partial B_r$ and $y \in Tx$ we have $(y, x) \neq \| x \|^2$.*

Proof : (i) $\Rightarrow$ (ii). Let $x \in \partial B_r$ and $\{e_i\}$ be an orthonormal basis of H, where $e_1 = x/r$. By assumption, the map T satisfies the condition (B), with respect to the orthonormal basis $\{e_i\}$, and $x \in \partial_1 B_r = \{x \in \partial B_r : (x, e_1)^2 = r^2\}$. Hence $(y, x) \neq \| x \|^2$ for each $y \in Tx$.

(ii) $\Rightarrow$ (i). Let $\{e_i\}$ be an arbitrary orthonormal basis of H. If T did not satisfy condition (B) with respect to $\{e_i\}$, then for all $n \in N$ we could find $x \in \partial_n B_r$ and $y \in Tx$ such that $(y, e_j) = (x, e_j)$ for $j = 1, \ldots, n$, hence

$$(y, x) = \| x \|^2, \text{ contradicting (ii).}$$

Theorem 3. 1.

If the map $T : B_r \rightarrow H$ is bounded and satisfies the following condition :

(b) $\begin{cases} \textit{There exists a real number } k > 0 \textit{ and an element } a \in H \textit{ such} \\ \textit{that, for all } x \in \partial B_r \textit{ and } y \in Tx, \textit{ one of the following} \\ \textit{conditions hold :} \\ \text{(I) } (x, y) \neq \| x \|^2 ; \\ \text{(II) } | (a, x-y) | \geqslant k ; \end{cases}$

then the map T satisfies the condition (A), with respect to every

Orthonormal basis $\{e_i\}$ *in H.*

Proof : Let $\{e_i\}$ be an orthonormal basis in H. Since T is bounded, there exists $n_0 \in N$ such that

$$\left| \sum_{j=n_0}^{\infty} (a, e_j)(y, e_j) \right| < k \qquad \text{for all } y \in TB_r,$$

Now, if the map T did not satisfy the condition (A), with respect to $\{e_i\}$, then for all $n \in N$ we could find $x \in \partial_n B_r$ and $y \in Tx$ such that

$$(x, e_j) = (y, e_j) \quad \text{for } j = 1, .., n.$$

If x verifies condition I) of (b), this contradicts Proposition 3. 1. On the other hand, if x verifies condition II) of (b), this gives a contradiction for $n \geqslant n_o$, since

$$k \leqslant |(a, x-y)| = \left| \sum_{j=m+1}^{\infty} (a, e_j)(y, e_j) \right| < k .$$

Remark 3. 1.

If $T : B_r \to H$ is a w. u. s. c. map which satisfies the condition (b) of Theorem 3. 1, then degree of Definition 3.1 is well defined with respect to every orthonormal basis [e_i] in H.

Remark 3. 2.

The class of singlevalued weakly continuous maps which satisfies the condition (b) of Theorem 3. 1 strictly includes the class for which the Canfora-Pacella degree theory applies (see [2] and [6]).

At last we give a fixed point theorem .

Theorem 3. 2.

Let $T : S \to H$ *be w. u. s. c. . Suppose that there exists a number* $\bar{n}$ *such that , for all* $n \geqslant \bar{n}$ *, the condition* $(y , x) \leqslant \| x \|^2$ *holds for any* $x \in \partial_n B_r$ *and* $y \in Tx$ *. Then T has a fixed point in* B_r *.*

Proof . We consider the homotopy $H(t, x) = t(I - T)x$. Then the proof of the theorem follows from the proposition 1·2.

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ON CERTAIN OPERATIONAL GENERATING AND RECURRENCE RELATIONS †

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ABSTRACT

In this paper we have deduced certain operational generating and operational recurrence relations for the polynomials sets, involving the operators $R^{(\alpha)}_{n,k}$ and $S^{(n)}_{\alpha,k}$ defined below. We have also given (as an illustration) the generating and recurrence relations for the generalized Rice polynomials .

1. INTRODUCTION

In the recent literature several authors have used various combinations of the differential operator to get generating and recurrence relations . For example, Mittal* ([3], [4]) has given certain operational

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* For a systematic presentation of various classes of generating functions that are rather easily derivable consequences of Lagrange's expansion theorem, see Chapter 7 of the latest work on the subject by Srivastava and Manocha ([8] , p. 354 *et seq.*) .

generating and recurrence relations involving the differential operator $T_k = x(k + xD)$, where k is a constant, for the polynomial sets; and, as a sequel, he has given some characterizations for special functions. He has also given the generating and recurrence relations for the generalized Laguerre, Hermite, Bessel, and Jacobi polynomials (see also Srivastava and Buschman [7] for a unified study of these classes of generating functions).

In this paper an attempt has been made to derive the operational generating and recurrence relations with the help of the operators $R^{(\alpha)}_{n,k}$ and $S^{(n)}_{\alpha,k}$ which are the combinations of the finite difference operators Δ_α ahd ∇_α, defined by (for $n=1,2,\ldots$)

$$\Delta_\alpha \left\{ f(\alpha) \right\} = f(\alpha+1) - f(\alpha),$$

$$\Delta^n_\alpha \left\{ f(\alpha) \right\} = \Delta_\alpha \left\{ \Delta^{n-1}_\alpha f(\alpha) \right\} \tag{1. 1}$$

and

$$\nabla_\alpha \left\{ f(\alpha) \right\} = f(\alpha) - f(\alpha-1),$$

$$\nabla^n_\alpha \left\{ f(\alpha) \right\} = \nabla_\alpha \left\{ \nabla^{n-1}_\alpha f(\alpha) \right\}, \tag{1. 2}$$

respectively.

In § 4, we obtain certain generating and recurrence relations for the generalized Rice polynomials [2].

2. Operational, Generating And Recurrence Relations

Consider the operator

$$R_{n,k}^{(\alpha)} = \prod_{j=1}^{n} [(\alpha - j) k + (\alpha - j)(\alpha - j + 1) \triangle_\alpha]$$

$$\equiv (-)^n (1 - \alpha)_n \prod_{j=1}^{n} [\alpha \triangle_\alpha + k + j - 1], \tag{2.1}$$

where k is constant and n a positive integer.

It can be proved easily, that

$$R_{n,k}^{(\alpha)} \{ (\alpha)_r \} = (k + r)_n (\alpha - n)_{n+r} \tag{2.2}$$

and

$$R_{n,k+mn}^{(\alpha)} \left\{ {}_{C+1}F_D \left[\begin{matrix} \alpha, (c); x \\ (d); \end{matrix} \right] \right\}$$

$$= (\alpha-n)_n (k+mn)_n \, {}_{C+2}F_{D+1} \left[\begin{matrix} \alpha, (c), k+(m+1)n; x \\ (d), k+mn; \end{matrix} \right] \tag{2.3}$$

Consider the function $G_s^{(\alpha)}(x, y)$, defined by

$$G_s^{(\alpha)} (x, y) = \sum_{r=0}^{\infty} (\alpha)_r A_r (x, s) \frac{y^r}{r!} , \tag{2.4}$$

where, $A_r(x, s)$ are independent of α and n.

Using the Lagrange expansion formula [5]

$$\prod_{n=0}^{\infty} \alpha (\alpha + mn + 1)_{n-1} \frac{t^n}{n!} = (1 + v)^\alpha \tag{2.5}$$

and the well-known result [5]

$$\frac{(1 + v)^{\alpha+1}}{1-mv} = \sum_{n=0}^{\infty} \binom{\alpha + (m + 1)n}{n} t^n \tag{2.6}$$

where $v = t(1 + v)^{m+1}$, m being a constant. The following results can be proved easily;

$$(\alpha - n)\, R^{(\alpha)}_{n-1,\, k+mn+1}\Big[\, k\, G^{(\alpha)}_s(x, y) + \alpha \triangle_\alpha\, G^{(\alpha)}_s(x, y)\,\Big]$$

$$= R^{(\alpha)}_{n,\, k+mn+1}\Big\{\, G^{(\alpha)}_s(x, y)\Big\} - n\,(m+1)\,(\alpha - n)\, R^{(\alpha)}_{n-1,\, k+mn+1}\Big\{\, G^{(\alpha)}_s(x.y)\Big\}, \tag{2.7}$$

$$(\alpha - n)\, R^{(\alpha)}_{n-1,\, k+mn+1}\Big[\, \mathrm{k}\, G^{(\alpha)}_s(x, y) + \alpha \triangle_\alpha\, G^{(\alpha)}_s(x, y)\Big]$$

$$= R^{(\alpha)}_{n,\, k+mn}\Big\{\, G^{(\alpha)}_s(x,y)\Big\} - mn(\alpha - n)\, R^{(\alpha)}_{n-1,\, k+mn+1}\Big\{\, G^{(\alpha)}_s(\mathrm{x}, y)\Big\}, \tag{2 8}$$

$$(\alpha - n)\, \triangle_\alpha\, [\, R^{(\alpha)}_{n,\, k+mn}\ (\, G^{(\alpha)}_s(x, y)\,]$$

$$= \frac{1}{\alpha - n - 1}\, R^{(\alpha)}_{n+1,\, k+mn}\ \{\, G^{(\alpha)}_s(x, y)\,\} -$$

$$-\,(k + mn)\, R^{(\alpha)}_{n,\, k+mn}\ \{\, G^{(\alpha)}_s(x, y)\,\}, \tag{2 9}$$

$$(\alpha - n)\, \triangle_\alpha\ [\, R^{(\alpha)}_{n,\, k+mn}\ (\, G^{(\alpha)}_s(x, y)\,]$$

$$= \frac{1}{\alpha - n - r}\, R^{(\alpha)}_{n+1,\, k+mn-1}\ \{\, G^{(\alpha)}_s(x, y)\,\} -$$

$$-\,(k + mn - n - 1)\, R^{(\alpha)}_{n,\, k+mn}\ \{\, G^{(\alpha)}_s(x, y)\,\}, \tag{2.10}$$

$$R^{(\alpha)}_{n,\, k+mn+1}\ \{\, G^{(\alpha)}_s(x, y)\,\}$$

$$= n\,(\alpha - n) R^{(\alpha)}_{n-1,\, k+mn+1}\ \{\, G^{(\alpha)}_s(x, y)\,\} + K^{(\alpha)}_{n,\, k+mn}\ \{\, G^{(\alpha)}_s(x, y)\,\}, \tag{2.11}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \Gamma(\alpha - n + 1)\, R^{(\alpha)}_{n-1,\, k+mn+1}$$

$$\left[k\, G_s^{(\alpha)}(x, y) + \sum_{p=1}^{\infty} (\alpha)_p \triangle_\alpha^p\, G_s^{(\alpha)}(x, y)\right]$$

$$= \Gamma(\alpha)(1+v)^k \sum_{r=0}^{\infty} \frac{(\alpha)_r\, A_r(x, s)\left[k + \sum_{p=1}^{\infty} (-)^p (-r)_p\right] (y(1+v))^r}{(k+r)\, r!} \tag{2.12}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \Gamma(\alpha - n + 1)\, R_{n-1,\, k+mn+1}^{(\alpha)}$$

$$\left[k\, G_s^{(\alpha)}(x, y) + \alpha \triangle_\alpha\, G_s^{(\alpha)}(x, y)\right]$$

$$= \Gamma(\alpha)(1+v)^k\, G_s^{(\alpha)}(x, y(1+v)), \tag{2.13}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \Gamma(\alpha - n)\left[R_{n,k+mn+1}^{(\alpha)}\left(G_s^{(\alpha)}(x, y)\right)\right.$$

$$\left. - n(m+1)(\alpha - n)\, R_{n-1,\, k+mn+1}^{(\alpha)}\left(G_s^{(\alpha)}(x, y)\right)\right]$$

$$= \Gamma(\alpha)(1+v)^k\, G_s^{(\alpha)}(x, y(1+v)), \tag{2.14}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \Gamma(\alpha - n)\left[R_{n,\, k+mn}^{(\alpha)}\left(G_s^{(\alpha)}(x, y)\right)\right.$$

$$\left. - mn(\alpha - n)\, R_{n-1,\, k+mn+1}^{(\alpha)}\left(G_s^{(\alpha)}(x, y)\right)\right]$$

$$= \Gamma(\alpha)(1+v)^k\, G_s^{(\alpha)}(x, y(1+v)), \tag{2.15}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \Gamma(\alpha - n)\, R_{n,\, k+mn+1}^{(\alpha)}\left\{G_s^{(\alpha)}(x, y)\right\} =$$

$$=\frac{\Gamma(a)\,(1+v)^{k+1}}{(1-mv)}\;G_s^{(\alpha)}\,(x,\,y\,(1+v)), \tag{2.16}$$

where, $v = t\,(1+v)^{m+1}$.

3. THE OPERATOR $S_{\alpha,k}$

In this section we deduce some operational generating and recurrence relations for the polynomial sets $\{ G_s^{(\alpha)}(x, y) \}$. But this time we shall make the use of the operator $S_{\alpha,\,k}$, defined by

$$S_{\alpha,\,k} = \frac{1}{1-\alpha}\,[k + (1 - a - k)\,\nabla_\alpha\,]. \tag{3.1}$$

where the operator ∇_α is given by (1.2) and k is a constant. It is easily seen, by induction hypothesis, that

$$S^n_{\alpha,k}\,\{ (\alpha)_r \} = (-)^n\,(k - r)_n\,(\alpha)_{r-n} \tag{3.2}$$

where n is a positive integer and r is any integer.

Also, we have

$$S^n_{\alpha,k} = \frac{1}{(1-a)_n}\prod_{j=1}^{n}\,[\,(1 - \alpha - k)\,\nabla_\alpha + k + j - 1\,]. \tag{3.3}$$

One can easily, show that

$$S^{n-1}_{\alpha+n,k+mn+1}\,[\,(a + n)\,\Delta_\alpha\,G_s^{(\alpha+n)}(x, y) - k\,G_s^{(\alpha+n)}(x, y)$$

$$= (a + n)\,S^n_{\alpha+n+1,\,k+mn+1}\left\{ G_s^{(\alpha+n+1}(x, y) \right\}$$

$$+ n\,(m + 1)\,S^{n-1}_{\alpha+n,\,k+mn+1}\left\{ G_s^{(\alpha+n)}(x, y) \right\}, \tag{3.4}$$

$$S^{n-1}_{\alpha+n,\ k+mn+1}\left[(\alpha+n)\,\triangle_\alpha\ \ G_s^{(\alpha+n)}(x,y)-k\ \ G_s^{(\alpha+n)}(x,y)\right]$$

$$=(\alpha+n)\ S^{n}_{\alpha+n+1,\ k+mn}\left\{G_s^{(\alpha+n+1)}(x,y)\right\}$$

$$+\,mn\,S^{n-1}_{\alpha+n,\ k+mn+1}\left\{G_s^{(\alpha+n)}(x,y)\right\}, \tag{3.5}$$

$$(\alpha+n)\,\triangle_\alpha\left[S^{n}_{\alpha+n,\ k+mn}\left(G_s^{(\alpha+n)}(x,y)\right)\right]$$

$$=(\alpha+n)\,S^{n+1}_{\alpha+n+1,\ k+mn-1}\left\{G_s^{(\alpha+n+1)}(x,y)\right\}$$

$$+\,(k+mn)\ S^{n}_{\alpha+n,\ k+mn}\left\{G_n^{(\alpha+n)}(x,y)\right\}, \tag{3.6}$$

$$(\alpha+n)\ \triangle_\alpha\left[S^{n}_{\alpha+n,\ k+mn}\left(G_s^{(\alpha+n)}(x,y)\right)\right]$$

$$=(\alpha+n)\ S^{n+1}_{\alpha+n+1,\ k+mn-1}\left\{G_s^{(\alpha+n+1)}(x,y)\right\}$$

$$+\,(k+mn-n-1)\ S^{n}_{\alpha+n,\ k+mn}\left\{G_s^{(\alpha+n)}(x,y)\right\}, \tag{3.7}$$

$$(\alpha+n-1)\,S^{n}_{\alpha+n,\ k+mn+1}\left\{G_s^{(\alpha+n)}(x,y)\right\}$$

$$=(\alpha+n-1)\ \ S^{n}_{\alpha+n,\ k+mn}\left\{G_s^{(\alpha+n)}(x,y)\right\}$$

$$-\,n\ S^{n-1}_{\alpha+n-1,\ k+mn+1}\left\{G_s^{(\alpha+n-1)}(x,y)\right\}, \tag{3.8}$$

$$\sum_{n=0}^{\infty}\frac{t^n\,(\alpha)_n}{n!}\quad S^{n-1}_{\alpha+n,\ k+mn+1}\Bigg[(\alpha+n)\,\triangle_\alpha\,G_s^{(\alpha+n)}\ \ (x,y)$$

$$-\,k\ \ G_s^{(\alpha+n)}(x,y)\Bigg]=\alpha\,(+v)^{k}\,G_s^{(\alpha+1)}(x,y/(l+v)), \tag{3.9}$$

$$\sum_{n=0}^{\infty} \frac{t^n (\alpha)_n}{n!} \Big[(\alpha + n) S^n_{\alpha+n+1, k+mn+1} \left\{ G_s^{(\alpha+n+1)}(x, y) \right\}$$

$$+ n(m+1) \; S^{n-1}_{\alpha+n, k+mn+1} \left\{ G_s^{(\alpha+n)}(x, y) \right\} \Big]$$

$$= \alpha (1+v)^k \; G_s^{(\alpha+1)} (x, y/(1+v)), \tag{3.10}$$

$$\sum_{n=0}^{\infty} \frac{t^n (\alpha)_n}{n!} \Big[(\alpha + n) \; S^n_{\alpha+n+1, k+mn} \left\{ G_s^{(\alpha+n+1)}(x, y) \right\}$$

$$+ m \; n \; S^{n-1}_{\alpha+n, k+mn+1} \left\{ G_s^{(\alpha+n)} (x, y) \right\} \Big]$$

$$= \alpha (1+v)^k \; G_s^{(\alpha+1)} (x, y/(1+v)), \tag{3.11}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n (\alpha)_n}{n!} \; S^n_{\alpha+n, k+mn+1} \left\{ G_s^{(\alpha+n)}(x, y) \right\}$$

$$= \frac{(1+v)^{k+1}}{(1-mv)} \; G_s^{(\alpha)} (x, y/(1+v)) \tag{3.12}$$

where,

$$v = -t(1+v)^{m+1} \; .$$

4. APPLICATIONS

In this section, we give certain generating and recurrence relations involving generalized Rice's polynomial [2]

$$H_n^{(\alpha, \beta)}(p, q; x) = \frac{(1+\alpha)_n}{n!} \; {}_3F_2 \left[\begin{matrix} -n, 1+\alpha+\beta+n, p; x \\ 1+\alpha, q; \end{matrix} \right]$$

with the help of the operational relations obtained in § 2.

From the equation (2.11), we have

$$R_{n,\; k+mn+1}^{(\alpha+\beta+1)} \left\{ G_s^{(\alpha+\beta+1)} (x, y) \right\}$$

$$= n(\alpha+\beta-n+1) \quad R_{n-1,\; k+mn+1}^{(\alpha+\beta+1)} \left\{ G_s^{(\alpha+\beta+1)} (x, y) \right\}$$

$$+ R_{n,\; k+mn}^{(\alpha+\beta+1)} \left\{ G_s^{(\alpha+\beta+1)} (x, y) \right\} \tag{4.1}$$

where,

$$R_{n,\; k}^{(\alpha+\beta+1)} \equiv \prod_{j=1}^{n} \left[(\alpha + \beta + 1 - j) k + (\alpha + \beta + 1 - j) (\alpha + \beta - j + 2) \triangle \beta \right].$$

Let,

$$G_s^{(\alpha+\beta+1)} (x, y) = (- x)^{-s} \; P_s^{(\alpha-s,\beta)} (1 - 2 x) , \tag{4.2}$$

where $P_s^{(\alpha,\beta)} (x)$ is the Jacobi polynomial [6].

It is obvious that

$$\triangle_\beta \left\{ G_s^{(\alpha+\beta+1)} (x, y) \right\} = G_{s-1}^{(\alpha+\beta+2)} (x, y) .$$

Using (2.2) , one can easily show that

$$R_{n,k}^{(\alpha+\beta+1)} \left\{ P_s^{(\alpha-s,\beta)} (1 - 2 x) \right\}$$

$$= (\alpha + \beta - n + 1)_n \, (k)_n \, H_s^{(\alpha-s,\beta)} (k + n, k; x). \tag{4.3}$$

Putting the value of $G_s^{(\alpha+\beta+1)} (x, y)$ from (4.2) in (4.1) and making the use of (4.3), we get

$$(k + (m + 1)\, n)\ \ H_s^{(\alpha, \beta)}\,(k + (m + 1)\, n + 1,\ k + mn + 1;\ x)$$

$$= n\, H_s^{(\alpha, \beta)}\,(k + (m + 1)\, n,\ k + mn + 1;\ x)$$

$$+\ (k + mn)\ H_s^{(\alpha, \beta)}\,(k + (m + 1)\, n,\ k + mn;\ x). \tag{4.4}$$

Similarly, starting from (2.9), (2.10), (2.14), (2.15) and (2.16) and using (4.2), (4.3) we can easily derive the following formulae :

$$\Delta_\beta \Big[\ (\alpha + \beta - n + s + 1)_n\ H_s^{(\alpha, \beta)}\,(k + (m + 1)\, n,\ k + mn;\ x)\ \Big]$$

$$= (\alpha + \beta - n + s + 2)_{n-1}$$

$$(k + (m + 1)\, n)\ \ H_s^{(\alpha, \beta)}\,(k + (m + 1)\, n + 1,\ k + mn,\ x)$$

$$-(\alpha + \beta - n + s + 2)_{n-1}\,(k + mn)\ \ H_s^{(\alpha,\beta)}(k + (m+1)n,\ k + mn; x),$$

$$\tag{4.5}$$

$$\Delta_\beta \Big[(\alpha + \beta - n + s + 1)_n\, H_s^{(\alpha,\beta}\ (k + (m + 1)\, n,\ k + mn;\ x)\Big]$$

$$= (\alpha + \beta - n + s + 2)_{n-1}\,(k + mn - 1)$$

$$H_s^{(\alpha,\beta}\ (k + (m + 1)\, n,\ k + mn - 1;\ x) - (\alpha + \beta - n + s + 2)_{n-1}$$

$$(k + mn - n - 1)\ \ H_s^{(\alpha,\beta)}\,(k + (m + 1)\, n,\ k + mn;\ x), \tag{4.6}$$

$$\sum_{n=0}^{\infty}\ \frac{t^n\, \Gamma(k + mn + n)}{n!\ \Gamma(k + mn + 1)}\Big[(k + (m + 1)\, n)\ H_s^{(\alpha,\beta)}\,(k + m + 1)n + 1,$$

$$k + mn + 1;\ x) - n\,(m + 1)\ H_s^{(\alpha,\beta)}\,(k + m + 1)n,\ k + mn + 1;\ x)\ \Big]$$

$$= \ (1 + v)^k\ P_s^{(\alpha,\beta}\,(1 - 2\,x\ (1 + v)\,), \tag{4.7}$$

$$\sum_{n=0}^{\infty} \frac{t^n \Gamma(k + mn + n)}{n! \, \Gamma(k + mn + 1)} \Big[k + mn) H_s^{(\alpha, \beta)} (k + (m + 1) n,$$

$$k + mn; x) - mn H_s^{(\alpha, \beta)} (k + (m + 1) n, k + mn + 1; x) \Big]$$

$$= (1 + v)^k P_s^{(\alpha, \beta)} (1 - 2 x (1 + v)) \qquad (4 8)$$

and

$$\sum_{n=0}^{\infty} t^n \binom{k + (m + 1) n}{n} H_s^{(\alpha, \beta)} (k + (m+1) n+1, k+mn+1; x)$$

$$= \frac{(1 + v)^{k+1}}{1 - v} P_s^{(\alpha, \beta} (1 - 2 x (1 + v)), \qquad (4.9)$$

where, $v = t (1 + v)^{m+1}$.

In a similar way more generating and recurrence relations can be obtained for generalized Rice's polynomials from (3.4) to (3.12).

We conclude this paper with the remark that in (2.4), if $y = 1$

and $A_r (x, s) = P_{s-r}^{(0)} (x)$, then

$$G_s^{(\alpha)} (x, 1) = \sum_{r=0}^{s} \frac{(\alpha)_r}{r!} P_{s-r}^{(0)} (x).$$

The polynomials $G_s^{(\alpha)} (x, 1)$, satisfy the functional relation

$$\Delta_\alpha \left\{ G_s^{(\alpha)} (x, 1) \right\} = G_{s-1}^{(\alpha+1)} (x, 1),$$

which has been studied by the author in his recent paper [1] . Thus we can get the operational generating and recurrence relations for the polynomials $G_s^{(\alpha)}(x, 1)$, which satisfy the above functional relation, directly from the results derived in § 2 and § 3 .

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SOME LINEAR AND BILINEAR GENERATING RELATIONS INVOLVING HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

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ABSTRACT

The aim of this paper is to establish several linear and bilinear generating relations involving hypergeometric functions of three variables. Some specializations, relevant to the present discussion, are also discussed .

1. INTRODUCTION

If we use the notation

$$(a)_n = a(a+1)(a+2)\,..\,(a+n-1)\,;\quad (a)_0 = 1,$$

where a is arbitrary and n a positive integer, then the generalized Horn functions of three variables, defined by Pandey [5, *p*. 115], Dhawan ([2, pp. 241 – 242] , [3, p, 43 (1. 1)]) , and Srivastava [9, p. 105 (3. 5)] , are as follows :

$G_A(\alpha, \beta, \beta'; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+p}(\beta')_n}{(\gamma)_{n+p-m}\, m!\, n!\, p!} x^m y^n z^p, \qquad (1.1)$$

$G_B(\alpha, \beta_1, \beta_2, \beta_3; \gamma; x, y. z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\gamma)_{n+p-m}\, m!\, n!\, p!} x^m y^n z^p, \qquad (1.2)$$

${}_3G_A^{(1)}(\alpha, \beta_1, \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_{m+p}}{(\gamma)_{n+p-m}\, m!\, n!\, p!} x^m y^n z^p, \qquad (1.3)$$

${}_3G_B^{(1)}(\alpha, \beta_1, \beta_2; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta_1)_m(\beta_2)_n}{(\gamma)_{n+p-m}\, m!\, n!\, p!} x^m y^n z^p, \qquad (1.4)$$

$G_C(\alpha, \alpha_1, \beta, \beta_1; \gamma; x, y, z)$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{p-m}(\alpha_1)_n(\beta)_{m+p}(\beta_1)_n}{(\gamma)_{n+p-m}\, m!\, n!\, p!} x^m y^n z^p, \qquad (1,5)$$

$G_C{}^*(\alpha, \beta, \beta_1; \gamma; x, y, z)$

$$\sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+p}(\beta)_{m+n}(\beta_1)_{n-p}}{(\gamma)_{m+n-p}\, m!\, n!\, p!} x^m y^n z^p. \qquad (1.6)$$

For the precise regions of convergence of each of these triple Gaussian hypergeometric series, see the latest work on the subject by Srivastava and Karlsson ([12], p. 87, Section 3.4).

In the present investigation we also require the following relations (See, e. g. Srivastava and Manocha [13]) :

$$P_{m+n}^{(\alpha-m-n,\ \beta-m-n)}(x) = \frac{(-\alpha-\beta)_{m+n}}{(m+n)!}\left(\frac{1-x}{2}\right)^{m+n}\left(\frac{x+1}{x-1}\right)^{\alpha}$$

$${}_2F_1\left(m-\alpha-\beta+n,\ -\alpha;\ -\alpha-\beta;\ \frac{2}{x+1}\right),\ (m,\ n=0,\ 1,\ 2,\ldots), \qquad (1.7)$$

$$\sum_{n=0}^{\infty}(u-1)^n\, P_n^{(\alpha-n,\ \beta+n)}\left(\frac{1+u}{1-u}\right){}_2F_1(-n,\ \delta-\gamma;\ \delta;\ t)$$

$$= \frac{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)}{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}\ t^{\alpha}\,(1-u+ut)^{-\beta}\ {}_2F_1\left(\beta;\ \gamma;\ \delta+\alpha;\ \frac{-ut}{1-u+ut}\right), \qquad (1.8)$$

$$(1-t)^{-1-\alpha}\ exp\left(\frac{-t(x+y)}{1-t}\right){}_0F_1\left(-;\ 1+\alpha;\ \frac{xyt}{(1-t)^2}\right)$$

$$= \sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}\ L_n^{(\alpha)}(x)\,L_n^{(\alpha)}(y)\,t^n, \qquad (1.9).$$

$$P_n^{(\alpha,\ \beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left(-n,\ 1+\alpha+\beta+n;\ 1+\alpha;\ \frac{1-x}{2}\right), \qquad (1.10)$$

$${}_2F_1(a,\ b;\ c;\ z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,\ c-b;\ c;\ z), \qquad (1.11)$$

$$G_A(\alpha,\ \beta_1,\ \beta_2;\ \alpha;\ vt,\ v(1-2t),\ v(1-2t))$$

$$= \sum_{n=0}^{\infty}\frac{n!}{(\beta_1+\beta_2)_n}(v-1)\ P_n^{(-n,\ \beta_1+\beta_2+n)}\left(\frac{1+v}{1-v}\right)$$

$$P_n^{(\beta_1+\beta_2-1,\ -\beta_2-n)}(1-2t), \qquad (1.12)$$

$$G_B(\alpha,\ \beta_1,\ \beta_2,\ \beta_3;\ \alpha;\ vt,\ v(2t-1),\ v(2t-1))$$

$$= \sum_{n=0}^{\infty}\frac{n!}{(1-\beta_2-\beta_3)_n}(v-1)^n\ P_n^{(-n,\ 1-\beta_2-\beta_3+n)}\left(\frac{1+v}{1-v}\right)$$

$$P_n^{(-\beta_2-\beta_3,\ \beta_1+\beta_2+\beta_3-1-n)}(1-2t), \tag{1.13}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^n \ {}_0F_1\left[\begin{matrix} -; \\ 1-\lambda-n; \end{matrix} -y\right] = (1-z)^{-\lambda}\ {}_0F_1\left[\begin{matrix} -; \\ 1-\lambda; \end{matrix} y(z-1)\right], \tag{I.I4}$$

$$\sum_{n=0}^{\infty} \frac{(\nu)_n}{n!}\ {}_{A+1}F_B(-n, (a); (b); x)\, t^n$$

$$= (1-t)^{-\nu}\ {}_{A+1}F_B\left(-\nu, (a); (b); \frac{xt}{t-1}\right), \tag{1.15}$$

where (1. 7), (1.8) are known results given by Srivastava ([10, *p*. 92(6.4)], [11, *p*. 28, (6.7)]), (1.9), (1.10), (1.11) are given in Rainville [6, *p*. 212, *p*. 254, *p*. 60], (1.12), (1.13), (1.14) are due to Shrivastava [7, *p*. 154, (3.1), (3.2)], [8, *p*. 131, (6. 3-37)], and (1.15) is given by Chaundy [1, *p*. 62, (25)].

2. BILINEAR GENERATING RELATIONS

We establish here the following bilinear generating relations :

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!}\ G_A\Bigg(\lambda, \alpha, \beta; \lambda-r; xy(1-z),$$

$$\frac{x(1-2y)}{1-z},\ \frac{x(1-2y)}{1-z}\Bigg) z^{\gamma}$$

$$= (1-z)^{\lambda-1} \sum_{n=0}^{\infty} \frac{n!}{(\alpha+\beta)_n} (x-1)^n\, P_n^{(-n,\ \alpha+\beta+n)}$$

$$\left(\frac{1+x}{1+x}\right) P_n^{(\alpha+\beta-1,\ -\beta-n)}(1-2y), \tag{2.1}$$

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} G_B (\lambda, \alpha, \beta, \gamma; \lambda - r ; x y (1 - z) ,$$

$$\left. \frac{x (2 y - 1)}{1 - z} , \frac{x (2 y - 1)}{1 - z} \right) z^r$$

$$= (1 - z)^{\lambda - 1} \sum_{n = 0}^{\infty} \frac{n!}{(1 - \beta - \gamma)_n} (x - 1)^n$$

$$P_n^{(- n, 1 - \beta - \gamma + n)} \left(\frac{1 + x}{1 - x} \right)$$

$$P_n^{(- \beta - \gamma, \alpha + \beta + \gamma - 1 - n)} (1 - 2 y), \qquad (2.2)$$

$$\sum_{r = 0}^{\infty} \frac{1}{(1 + \beta)_r \, r!} \, {}_3G_A^{(1)} (\alpha, - \beta - \gamma ; \alpha ; (1 - x)(1 - t),$$

$$\left. \frac{- t (x + y)}{1 - t} , t \right) \left(\frac{yt}{(1 - t)^3} \right)^r$$

$$= x^{\beta} (1 - t)^{1+2\beta} \sum_{n = 0}^{\infty} \frac{n!}{(1+ \beta)_n} L_n^{(\beta)} (x) L_n^{(\beta)} (y) t^n, \qquad (2.3)$$

$$\sum_{r = 0}^{\infty} \frac{(\beta_1)_r}{r!} {}_3G_B^{(1)} (\alpha, \beta_1 + r, \beta_2; \alpha; xy, x (1-2 y), x) (x (1-y))^r$$

$$= exp (x) \sum_{n = 0}^{\infty} \frac{n!}{(\beta_1 + \beta_2)_n} (x - 1)^n P_n^{(-n, \beta_1 + \beta_2 + n)}$$

$$\left(\frac{1 + x}{1 - x} \right) P_n^{(\beta_1 + \beta_2 - 1, - \beta_2 - n)} (1 - 2 y), \qquad (2.4)$$

To prove (2.1), consider

$$\Delta = \sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} \; G_A\left(\lambda, \alpha, \beta; \lambda - r; x\,y\,(1-z),\right.$$

$$\left. \frac{x\,(1-2\,y)}{1-z}, \frac{x\,(1-2\,y)}{1-z} \right) z^r .$$

On expressing G_A in series form, we have

$$\Delta = \sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} \sum_{m,n,p=0}^{\infty} \frac{(\lambda)_{n+p-m}\,(\alpha)_{m+p}\,(\beta)_n}{(\lambda-r)_{n+p-m}\,m!\,n!\,p!}$$

$$\left\{ x\,y\,(1-z) \right\}^m \left\{ \frac{x\,(1-2\,y)}{1-z} \right\}^{n+p} z^r .$$

Again using the results

$$(\alpha)_{-k} = \frac{(-1)^k}{(1-\alpha)_k}, \tag{2.5}$$

$$(\lambda)_r\,(\lambda+r)_s = (\lambda)_{r+s} = (\lambda)_s\,(\lambda+s)_r , \tag{2.6}$$

and

$$(1-x)^{-\lambda} = \sum_{i=0}^{\infty} \frac{(\lambda)_i}{i!} x^i , \tag{2.7}$$

we get

$$\Delta = (1-z)^{\lambda-1} \sum_{m,n,p=0}^{\infty} \frac{(\lambda)_{n+p-m}\,(\alpha)_{m+p}\,(\beta)_n}{(\lambda)_{n+p-m}\,m!\,n!\,p!} (x\,y)^m$$

$$\left\{ x\,(1-2\,y) \right\}^{n+p} .$$

Now applying the result (1.1), we obtain

$$\Delta = (1-z)^{\lambda-1}\; G_A\left(\lambda, \alpha, \beta; \lambda, xy, x\,(1-2\,y), x\,(1-2\,y)\right)$$

which, in light of (1.12), provides (2.1).

The proof of the formula (2.2) would run parallel to what we have obtained above in view of (1.2) and (1.13).

To prove (2.3), let

$$\tau = \sum_{r=0}^{\infty} \frac{1}{(1+\beta)_r\, r!}\; {}_3G_A^{(1)}\left(\alpha, -\beta-\gamma; \alpha; (1-x)(1-t), \frac{-t(x+y)}{1-t}, t\right)\left(\frac{yt}{(1-t)^3}\right)^r$$

$$= \sum_{r=0}^{\infty} \frac{1}{(1+\beta)_r\, r!} \sum_{m,n,p=0}^{\infty} \frac{(-\beta-r)_{m+p}}{m!\, n!\, p!}$$

$$\left\{(1-x)(1-t)\right\}^m \left\{\frac{-t(x+y)}{1-t}\right\}^n t^p \left(\frac{yt}{(1-t)^3}\right)^r.$$

Now applying the formulae

$$\sum_{i=0}^{\infty} \frac{x^i}{i!} = exp\,(x)\,, \tag{2.8}$$

and (2 5), we find that

$$\tau = exp\left(\frac{-t(x+y)}{1-t}\right) \sum_{m,p,r=0}^{\infty} \frac{(-\beta)_{-r}\,(-\beta-r)_{m+p}}{m!\; p!\; r!}$$

$$\left\{(1-x)(1-t)\right\}^m t^p \left(\frac{-yt}{(1-t)^3}\right)^r.$$

Again, using the results (2.6), (2.7) and (2.5)

$$\tau = (1-t)^{\beta}\; exp\left(\frac{-t(x+y)}{1-t}\right) \sum_{m,r=0}^{\infty} \frac{(-\beta)_m}{(1+\beta-m)_r\, m!\, r!}$$

$$(1 - x)^m \left(\frac{yt}{(1-t)^2} \right)^r$$

$$= (1-t)^\beta \ exp \left(\frac{-t(x+y)}{1-t} \right) \sum_{m=0}^{\infty} \frac{(-\beta)_m}{m!} (1-x)^m$$

$$_0F_1 \left[\begin{matrix} - ; & \\ 1+\beta-m; & \frac{yt}{(1-t)^2} \end{matrix} \right].$$

Now employing the result (1.14), we get

$$\tau = x^\beta (1-t)^\beta \ exp \left(\frac{-t(x+y)}{1-t} \right) {}_0F_1 \left[\begin{matrix} - ; & \\ 1+\beta; & \frac{xyt}{(1-t)^2} \end{matrix} \right]$$

which, in view of (1.9), yields (2.3) .

To prove (2.4), put

$$\phi = \sum_{r=0}^{\infty} \frac{(\beta_1)_r}{r!} \, {}_3G_B^{(1)}\left(\alpha, \beta_1 + r, \beta_2; \alpha; xy; x(1-2y), x \right) \{x(1-2y)\}^r$$

$$= \sum_{r=0}^{\infty} \frac{(\beta_1)_r}{r!} \sum_{m,n,p=0}^{\infty} \frac{(\beta_1 + r)_m (\beta_2)_n}{m!\, n!\, p!} (xy)^m$$

$$\{x(1-2y)\}^n x^p \{x(1-2y)\}^r .$$

Now applying the results (2.6) and (2.8),

$$\phi = exp(x) \sum_{m,n,r=0}^{\infty} \frac{(\beta_1)_{m+r} (\beta_2)_n}{m!\, n!\, r!} (xy)^m$$

$$\{x(1-2y)\}^{n+r}$$

$$= exp(x) \sum_{m,n=0}^{\infty} \sum_{r=0}^{n} \frac{(\beta_1)_{m+r} (\beta_2)_{n-r}}{m!\,(n-r)!\, r!} (xy)^m$$

$$\{x(1-2y)\}^n .$$

Then, on reversing the inner summation, we get

$$\phi = exp\,(x) \sum_{m,\, n = 0}^{\infty} \sum_{r = 0}^{n} \frac{(\beta_1)_{m+n-r}\,(\beta_2)_r}{m!\, r!\, (n-r)!}\,(x\,y)^m \{x\,(1-2\,y)\}^n$$

Now applying the relations (2.6), (2.5),

$$(n-k)! = \frac{(-1)^k\, n!}{(-n)_k}, \tag{2.9}$$

and Exercise 4 of Rainville [6, *p.* 69] , we obtain

$$\phi = exp\,(x) \sum_{m,\, n = 0}^{\infty} \frac{(\beta_1)_{m+n}\,(1-\beta_1-\beta_2-m-n)_n}{(1-\beta_1-m-n)_n\, m!\, n!}\,(xy)^m \{x\,(1-2\,y)\}^n$$

$$= exp\,(x) \sum_{n = 0}^{\infty} \sum_{m = 0}^{n} \frac{(\beta_1)_n\,(1-\beta_1-\beta_2-n)_{n-m}}{(1-\beta_1-n)_{n-m}\, m!\,(n-m)!}\,(x\,y)^m \{x\,(1-2\,y\}^{n-m}$$

which, on reversing the inner summation and simplifying, provides

$$\phi = exp\,(x) \sum_{n = 0}^{\infty} \sum_{m = 0}^{n} \frac{(\beta_1)_n\,(-n)_m\,(1-\beta_1-\beta_2-n)_m}{(1-\beta_1-n)_m\, m!\, n!}\,(x\,y)^{n-m} \{x\,(2\,y-1)\}^m .$$

Expressing the inner series as a Jacobi polynomial, using the result (1.7) (with $m = 0$) and (1.11), one finds

$$\phi = exp\,(x) \sum_{n = 0}^{\infty} \frac{(\beta_1+\beta_2)_n}{n!}\, {}_2F_1\!\left(-n, \beta_2; \beta_1+\beta_2; \frac{xy}{x\,(1-y)}\right) \{x\,(1-y)\}^n .$$

Finally, in the light of the results (1.15) (with $A = B = 1$) and (1.8), (2.4) is obtained .

3. LINEAR GENERATING RELATIONS

In this section we establish the following generating relations :

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} G_A(\alpha, \beta, \gamma; \lambda - r; x, y, z)\, t^r$$

$$= (1-t)^{\lambda-1} G_A\left(\alpha, \beta, \gamma; \lambda; \frac{x}{1-t}, y(1-t), z(1-t)\right), \quad (3.1)$$

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} G_B(\alpha, \beta, \gamma, \delta; \lambda - r; x, y, z)\, t^r$$

$$= (1-t)^{\lambda-1} G_B\left(\alpha, \beta, \gamma, \delta; \lambda; \frac{x}{1-t}, y(1-t), z(1-t)\right), \quad (3.2)$$

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} \, {}_3G_A^{(1)} (\alpha, \beta; \lambda - r; x, y, z) t^r$$

$$= (1-t)^{\lambda-1} \, {}_3G_A^{(1)} \left(\alpha, \beta; \lambda; \frac{x}{1-t}, y(1-t), z(1-t)\right), \quad (3.3)$$

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} \, {}_3G_B^{(1)} (\alpha, \beta_1, \beta_2; \lambda - r; x, y, z) t^r$$

$$= (1-t)^{\lambda-1} \, {}_3G_B^{(1)} \left((\alpha, \beta_1, \beta_2; \lambda; \frac{x}{1-t}, y(1-t), z(1-t)\right), \quad (3.4)$$

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} G_C (\alpha, \alpha_1, \beta_1, \beta_2; \lambda - r; x, y, z) t^r$$

$$= (1-t)^{\lambda-1}\ G_C\left(\alpha, \alpha_1, \beta_1, \beta_2; \lambda; \frac{x}{1-t}, y(1-t), z(1-t)\right), \tag{3.5}$$

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!}\ G_C^*(\alpha, \beta, \gamma; \lambda-r; x, y, z)t^r$$

$$= (1-t)^{\lambda-1}\ G_C^*\left(\alpha, \beta, \gamma; \lambda; x(1-t), y(1-t), \frac{z}{1-t}\right). \tag{3.6}$$

Derivation of (3.1) : consider

$$\Omega = \sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!}\ G_A(\alpha, \beta, \gamma; \lambda-r; x, y, z)t^r.$$

On expressing G_A in series form, we get

$$\Omega = \sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!} \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+p}(\gamma)_n}{(\lambda-\gamma)_{n+p-m}\, m!\, n!\, p!} x^m y^n z^p t^r.$$

Again using the results (2.5) and (2.6), we have

$$\Omega = \sum_{m,n,p,r=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+p}(\gamma)_n(1-\lambda-n-p+m)_r}{(\lambda)_{n+p-m}\, m!\, n!\, p!\, r!} x^m y^n z^p t^r.$$

Now applying (2.7), we find that

$$\Omega = (1-t)^{\lambda-1} \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{n+p-m}(\beta)_{m+p}(\gamma)_n}{(\lambda)_{n+p-m}\, m!\, n!\, p!} \left(\frac{x}{1-t}\right)^m$$

$$\left\{ y(1-t) \right\}^n \left\{ z(1-t) \right\}^p$$

which, in view of (1.1), provides (3.1).

Proceeding on similar lines as above results (3.2) to (3.6) can be derived.

4. SPECIAL CASES

(i) On taking $\beta=0$ in (2.1) and simplifying, we obtain

$$G_1\,(\alpha, \lambda, 1-\lambda; -xy, x\,(2y-1)\,)$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(\alpha)_n} (x-1)^n\, P_n^{(-n,\ \alpha+n)} \left(\frac{1+x}{1-x}\right) P_n^{(\alpha-1,\ -n)}\,(1-2y), \tag{4.1}$$

where G_1 is Horn's function [4, *p*. 224 (10)] .

(ii) In (2.1) putting $y=0$, letting $\alpha=0$ and solving, we find that

$${}_2F_1\left[\begin{matrix}\lambda, \beta; \\ \lambda;\end{matrix}\ x\right] = \sum_{n=0}^{\infty} (x-1)^n\, P_n^{(-n,\ \beta+n)}\left(\frac{1+x}{1-x}\right). \tag{4.2}$$

(iii) In (2.2) setting $y=0$, putting $\alpha=0$, replacing $-x$ by x and simplifying, we get

$$F_1\,(\lambda, \beta; \gamma; x, x) = \sum_{n=0}^{\infty} (1-x)^n\, P^{(-n,\ 1-\beta-\gamma+n)} \left(\frac{1-x}{1+x}\right), \tag{4.3}$$

where F_1 is Appell's double hypergeometric function [5, *p*. 265].

(iv) Changing y to y/β_1 and letting $\beta_1 \to \infty$, (3.4) yields

$$\sum_{r=0}^{\infty} \frac{(1-\lambda)_r}{r!}\ {}_3G_A^{(2)}\,(\alpha, \beta_2; \lambda-r; x, y, z)\,t^r$$

$$= (1-t)^{\lambda-1}\ {}_3G_A^{(2)}\left(\alpha, \beta_2; \lambda; \frac{x}{1-t}, y\,(1-t), z\,(1-t)\right), \tag{4.4}$$

where ${}_3G_A{}^{(2)}$ is given earlier [2, pp. 241—242] .

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ON CERTAIN POLYNOMIALS ASSOCIATED WITH THE MULTIVARIABLE *H*-FUNCTION

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ABSTRACT

In an attempt to unify various bilateral generating functions obtained earlier by Srivastava and Panda [6], Raina [2], and Srivastava and Raina [10], we study here a few new sets of polynomials associated with the multivariable *H*-function of Srivastava and Panda ([6]; see also [7] and [8]), and give certain theorems concerning the generating functions of these polynomials. In the sequel we also show that these theorems can be applied to yield several bilateral generating functions for some polynomial sets .

1. INTRODUCTION AND PREREQUISITES

For convenience, let $(c_j, \gamma_j)_{1,p}$ and $(a_j; \alpha_j', \ldots, \alpha_j^{(s)})_{1,p}$ abbreviate the p member arrays $(c_1, \gamma_1), \ldots, (c_p, \gamma_p)$ and $(a_1; \alpha_1', \ldots, \alpha_1^{(s)}), \ldots, (a_p; \alpha_p', \ldots, \alpha_p^{(s)})$, $p \geqslant 0$ respectively . The array being empty if $p = 0$. Then we recall the definition of multivariable *H*-function introduced by Srivastava and Panda [6, p. 271, eq, (4 1)] in the following form :

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$$(1.1)\quad H^{(1)}\begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix} = H^{\,0,\, n:\, m',\, n';\dots;\, m^{(s)},\, n^{(s)}}_{\,p,\, q:\, p',\, q';\dots;\, p^{(s)},\, q^{(s)}} \left[\begin{matrix} z_1 \\ \vdots \\ z_s \end{matrix}\right| \begin{matrix} (a_j;\ \alpha_j',\dots,\alpha_j^{(s)})_{1,\,p}: (c_j',\gamma_j')_{1,\,p'};\dots;(c_j^{(s)},\gamma_j^{(s)})_{1,\,p^{(s)}} \\ (b_j;\ \beta'_j,\dots,\beta_j^{(s)})_{1,\,q}: d_j',\delta_j')_{1,\,q'};\dots;(d_j^{(s)})\ \delta_j^{(s)})_{1,\,q^{(s)}} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L_1}\cdots\int_{L_s}\Phi(\zeta_1,\dots,\zeta_s)\prod_{i=1}^{s}\left\{\theta_i(\zeta_i)\, z_i^{\zeta_i}\, d\zeta_i\right\},$$

where $\omega = \sqrt{-1}$,

$$(1.2)\quad \Phi(\zeta_1,\dots,\zeta_s) = \prod_{j=1}^{n}\Gamma\left(1 - a_j + \sum_{i=1}^{s}\alpha_j^{(s)}\zeta_i\right)$$

$$\cdot\left[\prod_{j=1}^{q_i}\Gamma\left(1 - b_j + \sum_{i=1}^{s}\beta_j^{(i)}\zeta_i\right)\prod_{j=n+1}^{p}\Gamma\left(a_j - \sum_{i=1}^{s}\alpha_j^{(i)}\zeta_i\right)\right]^{-1},$$

$$(1.3)\quad \theta_i(\zeta_i) = \prod_{j=1}^{m^{(i)}}\Gamma(d_j^{(i)} - \delta_j^{(i)}\zeta_i)\prod_{j=1}^{n^{(i)}}\Gamma(1 - c_j^{(i)} + \gamma_j^{(i)}\zeta_i)$$

$$\cdot\left[\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma(1 - d_j^{(i)} + \delta_j^{(i)}\zeta_i)\right.$$

$$\left.\cdot\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(c_j^{(i)} - \gamma_j^{(i)}\zeta_i)\right]^{-1}\ \forall\ i\in\{1,\dots,s\},$$

an empty product is interpreted as unity, the coefficient $\alpha_j^{(i)}$, $j = 1, 2, \dots, p$; $\beta_j^{(i)}$, $j = 1, 2, \dots, q$; $\gamma_j^{(i)}$, $j = 1, 2, \dots, p^{(i)}$; $\delta_j^{(i)}$, $j = 1, 2, \dots, q^{(i)}$; $i = 1, 2, \dots, s$ are positive real numbers and $n, p, q, m^{(i)}, n^{(i)}, p^{(i)}$ and $q^{(i)}$ are integers such that $0 \leqslant n \leqslant p$, $0 \leqslant m^{(i)} \leqslant q^{(i)}$; $q \geqslant 0$, $0 \leqslant n^{(i)} \leqslant p^{(i)}$, $i = 1, 2, \dots, s$. The multiple integral (1.1) converges absolutely if

$$(1.4)\quad U_i > 0 \text{ and } \left|\arg z_i\right| < \frac{1}{2}\pi U_i,$$

where

$$(1.5)\quad U_i = -\sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n^{(i)}} \gamma_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}} \gamma_j^{(i)} + \sum_{j=1}^{m^{(i)}} \delta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}} \delta_j^{(i)},$$

$$i \in \{ 1, 2. \ldots, s \}.$$

Again, for convenience, let

$$H\left[\begin{array}{c|c} z_1 & (\xi_j; \mu_j', \ldots, \mu_j^{(s)})_{1,u} \\ \vdots & \\ z_s & (\eta_j; \nu_j', \ldots, \nu_j^{(s)})_{1,v} \end{array}\right]$$

denote the multivariable H-function

$$(1.6)\quad H^{0,\, n+u:\, m',\, n';\, \ldots;\, m^{(s)},\, n^{(s)}}_{p+u,\, q+v:\, p',\, q';\, \ldots;\, p^{(s)},\, q^{(s)}} \left[\begin{array}{c|c} z_1 & (\xi_j; \mu_j', \ldots, \mu_j^{(s)})_{1,u}, \\ \vdots & \\ z_s & (\eta_j; \nu_j', \ldots, \nu_j^{(s)})_{1,v}, \end{array}\right.$$

$$\left.\begin{array}{l} (a_j; \alpha_j', \ldots, \alpha_j^{(s)})_{1,p}; (c_j', \gamma_j')_{1,p'}; \ldots; (c_j^{(s)}, \gamma_j^{(s)})_{1,p^{(s)}} \\ (b_j; \beta_j', \ldots, \beta_j^{(s)})_{1,q}; (d_j', \delta_j')_{1,q'}; \ldots; (d_j^{(s)}, \delta_j^{(s)})_{1,q^{(s)}} \end{array}\right].$$

In all the theorems discussed in this paper we shall assume the following :

(i) The restrictions on the parameters of the H-function and the convergence (and existence) conditions corresponding appropriately to the ones detailed above are satisfied by each of the various H-functions involved.

(ii) α_i's, β_i's, γ_i's, λ_i's, $i=1, 2, \ldots, r$ are arbitrary complex numbers independent of $n_1, \ldots, n_r$. Also $n_i \geqslant 0$, $q_i \in N$, where N is the set of natural numbers, $i=1, 2, \ldots, r$ and $\sigma_i^{(j)} > 0$, $j=1, 2, \ldots, s$; $i=1, 2, \ldots, r$.

(iii) (n_r) stands for the linear array $n_1, \ldots, n_r$ with similar interpretations for (α_r), (β_r) etc.

2. Generating functions for a general class of polynomials

Let f $[(z_r)]$ be a function of several complex variables $z_1, z_2, \ldots, z_r$ defined formally by the power series

$$(2.1)\quad f\,[(z_r)] = \sum_{k_1,\ldots,k_r=0}^{\infty} C\,[(k_r)] \prod_{i=1}^{r} \frac{z_i^{k_i}}{k_i!},$$

where the coefficients $C\,[(k_i)]$, $k_i \geqslant 0$, $i = 1,2,\ldots,r$ are arbitrary constants real or complex .

Also let a class of polynomials

$$(2.2)\quad Q_{(n_r);\,(q_r)}^{[(\alpha_r);\,(\beta_r)]}\,[(\lambda_r);\,(x_r);\,(y_s)] \text{ be defined by}$$

$$Q_{(n_r);\,(q_r)}^{[(\alpha_r);\,(\beta_r)]}\,[(\lambda_r);\,(x_r);\,(y_s)] = \sum_{k_1=0}^{[n_1/q_1]} \cdots \sum_{k_r=0}^{[n_r/q_r]} C\,[(k_r)]$$

$$\prod_{i=1}^{r} \left\{ (-n_i)_{q_i k_i} \frac{x_i^{k_i}}{k_i!} \right\}$$

$$.H \left[\begin{array}{c} y_1 \\ \vdots \\ y_s \end{array} \middle| \begin{array}{l} (-\alpha_i - (\beta_i + 1)\, n_i - \lambda_i k_i : \sigma_i', \ldots, \sigma_i^{(s)})_{1,r} \\ (-\alpha_i - \beta_i n_i - (\lambda_i + q_i)\, k_i : \sigma_i', \ldots, \sigma_i^{(s)})_{1,r} \end{array} \right].$$

Then our first multidimensional generating function is given by

Theorem 1. *With the function* f $[(z_r)]$ *defined* by (2.1), *let*

$$(2.3)\quad \theta\,[(n_r),\,(q_r);\,(\alpha_r),\,(\beta_r),\,(\gamma_r);\,(\lambda_r),\,(x_r),\,(y_s)]$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} C\,[(k_r)] \prod_{i=1}^{r} \left\{ \frac{\gamma_i}{\gamma_i + (\beta_i + 1)\, q_i k_i} \frac{x_i^{k_i}}{k_i!} \right.$$

$$.\begin{pmatrix} n_i + q_i k_i + \gamma_i / (\beta_1 + 1) \\ n_i \end{pmatrix}^{-1} \Bigg\}$$

$$.H \left[\begin{matrix} y_1 \\ \vdots \\ y_i \end{matrix} \middle| \begin{matrix} (\gamma_i - \alpha_i - \lambda_i k_i ; \sigma_i', \ldots, \sigma_i^{(s)})_{1,r} \\ (\gamma_i - a_i + n_i - \lambda_i k_i' ; \sigma_i', \ldots, \sigma_i^{(s)})_{1,r} \end{matrix} \right],$$

and also let

$$(2.4) \quad \phi\,[(x_r), (y_s); (v_r)] = \sum_{n_1,\ldots,n_r = 0}^{\infty} \theta\,[(n_r), (q_r); (\alpha_r), (\beta_r), (\gamma_r), (\lambda_r); (x_r), (y_s)] \prod_{i=1}^{r} \left\{ \frac{v_i^{n_i}}{n_i!} \right\},$$

Then

$$(2.5) \quad \sum_{n_1,\ldots,n_r=0}^{\infty} \prod_{i=1}^{r} \left(\frac{\gamma_i}{\gamma_i + (\beta_i + 1) n_i} \right) Q_{(n_r); (q_r)}^{[(a_r); (\beta_r)]} [(\lambda_r); (x_r), (y_s)] \cdot \prod_{i=1}^{r} \left(\frac{t_i^{n_i}}{n_i!} \right)$$

$$= \prod_{i=1}^{r} (1 + v_i)^{a_i} \, \phi \Bigg[x_1 (-v_1)^{q_1} (1 + v_1)^{\lambda_1}, \ldots, x_r (-v_r)^{q_r} (1 + v_r)^{\lambda_r},$$

$$y_1 \prod_{i=1}^{r} (1 + v_i)^{\sigma_i'}, \ldots, y_s \prod_{t=1}^{r} (1 + v_i)^{\sigma_i^{(s)}} ; \frac{-v_1}{1 + v_1}, \ldots, \frac{-v_r}{1 + v_r} \Bigg],$$

where v_i is a function of t_i defined by

$$v_i = t_i (1 + v_i)^{\beta_i + 1}, v_i(0) = 0,$$

Proof : On replacing $Q^{[(\alpha_r);(\beta_r)]}_{(n_r),(q_r)}[(\lambda_r);(x_r),(y_s)]$ by its equivalent series from its definition (2.2) with the H-funtcion expressed in terms of its Mellin-Barnes contour integral representation (1.1) in the left member of (2.5), changing the order of summation and integration which is assumed to be permissible and then applying a result due to Srivastava and Panda [9; Th. 3, p. 34], and finally interpreting the resulting expression by means of (1.1), we immediately arrive at the right member of (2.5) .

3. Two more classes of general polynomials

(i) The polynomials $T^{[(\alpha_r);(\beta_r)]}_{(n_r;(q_r)}[(\lambda_r);(x_r),(y_s)]$:

Associated with the power series (2.1), we may define a new set of general polynomials in the form

$$(3.1)\quad T^{[(\alpha_r);(\beta_r)]}_{(n_r);(q_r)}[(\lambda_r);(x_r),(y_s)] =$$

$$\sum_{k_1=0}^{[n_1/q_1]} \cdots \sum_{k_r=0}^{[n_r/q_r]} C[(k_r)] \prod_{i=1}^{r} \left\{ (-n_i)_{q_i k_i} \, . \, x_i^{k_i} \right\}$$

$$H\left[\begin{matrix} y_1 \\ \vdots \\ y_s \end{matrix}\left|\begin{matrix} (1-\alpha_i-(\beta_i+1)\,n_i-\lambda_i k_i;\,\sigma_i',\ldots,\sigma_i^{(s)})_{1,r} \\ (-\alpha_i-\beta_i n_i-(\lambda_i+q_i)\,k_i;\,\sigma_i',\ldots,\sigma_i^{(s)})_{1,r} \end{matrix}\right.\right\}.$$

The following theorem gives a generating function of type (2.5) with $\gamma_i = \alpha_i$, $i = 1,2,\ldots, r$ for this set of polynomials .

Theorem 2. *If we let*

$$(3.2)\quad \zeta[(x_r),(y_s)] = \sum_{k_1,\ldots,k_r=0}^{\infty} C[(k_r)] \prod_{i=1}^{r} \left\{ x_i^{k_i} \right\}$$

$$.H \left[\begin{array}{c|c} y_1 & (1-\alpha_i-(\beta_i+1)q_i k_i-\lambda_i k_i;\sigma_i',\ldots,\sigma_i^{(s)})_{1,r} \\ \vdots & \\ y_s & (-\alpha_i-(\beta_i+1)q_i k_i-\lambda_i k_i,\sigma_i',\ldots,\sigma_i^{(s)})_{1,r} \end{array} \right],$$

then

$$(3.3)\quad \sum_{n_1,\ldots,n_r=0}^{\infty} T_{(n_r);(q_r)}^{[(\alpha_r);(\beta_r)]} [(\lambda_r);(x_r),(y_s)] \prod_{i=1}^{r} \left(\frac{t_i^{n_i}}{n_i!} \right)$$

$$= \prod_{i=1}^{r} \left\{ (1+v_i)^{\alpha_i} \right\} \zeta [x_1 (-v_1)^{q_1} (1+v_1)^{\lambda_1}, \ldots,$$

$$x_r (-v_r)^{q_r} (1+v_r)^{\lambda_r},$$

$$y_1 \prod_{i=1}^{r} (1+v_i)^{\sigma_i'}, \ldots, y_s \prod_{i=1}^{r} (1+v_i)^{\sigma_i^{(s)}}],$$

The proof of Theorem 2 runs parallel to that of Theorem 1.

(ii) The polynomials $\Lambda_{(n_r);(\mu_r)}^{[(\alpha_r);(\beta_r)]} [(\lambda_r);(x_r),(y_s)]$:

Again with the meaning given earlier to $C[(k_r)]$, let us define another class of more general polynomials

$$(3.4)\quad \Lambda_{(n_r),(\mu_r)}^{[(\alpha_r);(\beta_r)]} [(\lambda_r);(x_r),(y_s)] =$$

$$\sum_{k_1,\ldots,k_r=0}^{\infty} C\,[(k_r)] \prod_{i=1}^{r} \left\{ \frac{x_i^{k_i}}{k_i!\,(n_i+1)_{\mu_i k_i}} \right\}$$

$$.H \left[\begin{array}{c|c} y_1 & (-\alpha_i-(\beta_i+1)n_i-\lambda_i k_i;\sigma_i',\ldots,\sigma_i^{(s)})_{1,r} \\ \vdots & \\ y_s & (-\alpha_i-\beta_i n_i-(\lambda_i-\mu_i)k_i;\sigma_i',\ldots,\sigma_i^{(s)})_{1,r} \end{array} \right],$$

and

$$(3.5)\quad \Lambda^{[(\alpha_r);\,(\beta_r)]}_{(-n_r);\,(\mu_r)}\,[(\lambda_r);\,(x_r),\,(y_s)]$$

$$= \sum_{k_1 \geqslant [\,n_1/\mu_1\,]}^{\infty} \cdots \sum_{k_r \geqslant [\,n_r/\mu_r\,]}^{\infty} C\,[(k_r)]$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{x_i^{k_i}}{k_i!\,(-n_i+1)_{\mu_i k_i}} \right\}$$

$$.H \left[\begin{matrix} y_1 \\ \vdots \\ y_s \end{matrix} \,\middle|\, \begin{matrix} (-\alpha_i + (\beta_i+1)\,n_i - \lambda_i k_i;\ \sigma_i', \ldots, \sigma_i^{(s)})_{1,r} \\ (-\alpha_i + \beta_i n_i - (\lambda_i - \mu_i)\,k_i;\ \sigma_i', \ldots, \sigma_j^{(s)})_{1,r} \end{matrix} \right]$$

By using the known result [9, Th, p. 38] , we can establish the following theorem giving a Laurent series expansion involving these polynomials .

Theorem 3. *in terms of the coefficients* $C\,[(k_r)]$, $k_i \geqslant 0$, $i = 1,2,\ldots,r$, *given by* (2.1), *let* θ *and* ϕ *be defined by equations* (2.3) *and* (2.4), *respectively, with* $q_i = -\mu_i$, $i = 1,\ldots,r$. *Then the polynomials defined by* (3.4) *and* (3.5) *are generated by*

$$(3.6)\quad \sum_{n_1,\ldots,n_r = -\infty}^{\infty} \prod_{i=1}^{r} \left\{ \frac{\gamma_i}{\gamma_i + (\beta_i+1)\,n_i}\, \frac{t_i^{n_i}}{n_i!} \right\}$$

$$\Lambda^{[(\alpha_r);\,(\beta_r)]}_{(n_r);\,(\mu_r)}\,[(\lambda_r);\,(x_r),\,(y_s)]$$

$$= \prod_{i=1}^{r} \left\{ (1+v_i)^{\alpha_i} \right\} \phi \Big[\, x_1\, v_1^{-\mu_1}\, (1+v_i)^{\lambda_1}, \ldots, x_r\, v_r^{-\mu_r}$$

$$(1+v_r)^{\lambda_r} \cdot\ y_1 \prod_{i=1}^{r} (1+v_i)^{\sigma_i'}\ ,\ ,$$

$$y_s \prod_{i=1}^{r} (1+v_1)^{\sigma_i^{(s)}}, \quad \frac{-v_1}{1+v_1}, \ldots, \frac{-v_r}{1+v_r}\Bigg].$$

4. SPECIAL CASES

(i) For $r = 1$, all these theorems reduce to the results given earlier by Srivastava and Raina [10].

(ii) Theorem 1, in its confluent case when $\gamma_i \to \infty$, $i = 1,2,\ldots,r$, reduces to the result :

$$(4.1) \qquad \sum_{n_1,\ldots,n_r=0}^{\infty} Q_{(n_r);\,(q_r)}^{[(\alpha_r);\,(\beta_r)]} [(\lambda_r); (x_r), (y_s)] \prod_{i=1}^{r} \left(\frac{t_i^{n_i}}{n_i!}\right)$$

$$= \prod_{i=1}^{r} \left\{ \frac{(1+v_i)^{\alpha_i+1}}{1-\beta_i v_i} \right\} f\Bigg[x_1(-v_1)^{q_1}(1+v_1)^{\lambda_1}, \ldots, x_r(-v_r)^{q_r}(1+v_r)^{\lambda_r} \Bigg]$$

$$.H^{(1)} \begin{bmatrix} y_1 \prod_{i=1}^{r} (1+v_i)^{\sigma_i'} \\ \vdots \\ y_s \prod_{i=1}^{r} (1+v_i)^{\sigma_i^{(s)}} \end{bmatrix}.$$

On making $\lambda_i \to 0$, $i = 1,2,\ldots, r$, the result (4.1) gives another important result which, on taking $r = 1$, $\alpha = \rho - 1$, $\beta = 0$, $p = n = q = 0$ and $\sigma_1 = \ldots = \sigma_s = h$, gives a known result of Raina [2, eq. (4.1), p. 303]. Moreover, each of Theorems 1 to 3 can be used to derive bilateral and multilateral generating functions associated with the multivariate H–functions by suitably specializing the coefficients $C[(k_r)]$, $k_i \geqslant 0$, $i = 1, 2,\ldots,r$.

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ON SOME SELF - SUPERPOSABLE FLOWS IN CONICAL DUCTS

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ABSTRACT

In finding out the exact analytic solutions of the basic equations of fluid dynamics, the principle of self-super-posability is of important use. In the present paper an attempt has been made to find some self-superposable motions of incomprerssible fluid in the conical ducts. No stress has been given on boundary conditions and the solutions thus determined containing a set of constants. Pressure distribution of some of such flows has also been discussed. The curves along which the vorticity of the flow becomes constant have also been attempted. The aim of the paper is to introduce a method for solving the basic equations of fluid dynamics in conical system of co-ordinates by using the property of self-superposability.

1. INTRODUCTION

It was shown by Ballabh ([2], [3]) that when a flow with velocity $\mathbf{q}$ satisfies the condition

$$curl\ (\mathbf{q} \times curl\ \mathbf{q}) = 0 \qquad \dots (1)$$

it becomes self-superposability. In the present paper the authors have attempted some fluid velocities for which ($\mathbf{q} \times curl\, \mathbf{q}$) = $\mathbf{p}$ (say) can be represented as the gradient of a scalar quantity . As such the flow will be self-superposable . Attention has been focussed on incompressible fluids only . For such fluids flowing in conical ducts, some velocities have been determined which satisfy the continuity condition and for which $\mathbf{p}$ can be represented as the gradient of a scalar quantity θ (say) . The solutions thus found must be of self-superposable type . Each will have a set of constants which can be determined by the boundary conditions .

It has further been shown by Ballabh [3] that if $\mathbf{q}_1$ and $\mathbf{q}_2$ are self-superposable flows and $\mathbf{q}_1 \pm \mathbf{q}_2$ are also self-super-posable then $\mathbf{q}_1$ and $\mathbf{q}_2$ are mutually superposable. By using this property some more self-superposable flows have also been determined pressure distributions for some of the flows have also been attempted . The curves along which the vorticity of the flow becomes constant have also been found .

In this paper a method will be introduced to solve the basic equations of fluid dynamics in conical co-ordinates by using the property of self-superposability , Mittal ([6].[7]), and Mittal, Thapaliyal and Agarwal [8] have recently solve the equations for different co-ordinate systems .

2. FORMULATION OF THE PROBLEM

Let

$$\mathbf{q} \times curl\, \mathbf{q} = \mathbf{p} \qquad \dots (2)$$

For incompressible fluids we have

$$div\, \mathbf{q} = 0 \qquad \dots (3)$$

Now, if a solution of equation (3) be found in such a way that $\mathbf{p}$

can be represented as the gradient of a scalar gradient of a scalar quantity (*say* θ) , *i, e.,*

$$\mathbf{p} = grad\,\theta \qquad \dots (4)$$

it will give a self-superposable flow It has been shown by Agarwal [1] that such solutions will also satisfy the equation of motion for a steady-flow . For determining a flow of liquid in conical duct let us consider the flow in conical co-ordinates (u, v, w) [4] .

If q_u, q_v, q_w be the components of $\mathbf{q}$ at any point (u, v, w) in conical co ordinates then equation (3) will become

$$\frac{1}{u^2}\frac{\partial}{\partial u}(u^2 q_u) + \frac{[(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}}}{u(v^2 - u^2)}$$

$$\frac{\partial}{\partial v}[(v^2 - w^2)^{\frac{1}{2}} q_v]$$

$$+ \frac{[(b^2 - w^2)(c^2 - w^2)]^{\frac{1}{2}}}{u(v^2 - w^2)}\frac{\partial}{\partial w}[(v^2 - w^2)^{\frac{1}{2}} q_w] = 0 \qquad \dots (5)$$

In order to make equation (5) integrable we may consider the following cases :

Case 1 : Let $q_u = 0$. In this case equation (5) will be satisfied by a solution

$$\left.\begin{aligned} q_u &= 0 \\ q_v &= \frac{AU(u)\;W(w)}{(v^2 - w^2)^{\frac{1}{2}}} \\ q_w &= \frac{BU_1(u)\;V(v)}{(v^2 - w^2)^{\frac{1}{2}}} \end{aligned}\right\} \qquad \dots (6)$$

where $U(u)$, $U_1(u)$ are the integrable functions of u; $V(v)$ and $W(w)$ the integrable functions of v and w respectively and A, B the constants .

For this fluid velocity, it can easily be shown that **p** can be represented by the gradient of a scalar quantity θ given by

$$\theta = \Bigg[\frac{A^2W^2U^2 + B^2V^2U_1^2)}{2(v^2 - w^2)} + \frac{A^2W^2}{(v^2 - w^2)} \int \frac{U^2}{u}\, du$$

$$+ \frac{B^2V^2}{(v^2 - w^2} \int \frac{U_1^2}{u}\, du + B^2U_1^2 \int \frac{VV'\, d}{(v^2 - w^2)}$$

$$- AB\, [(b^2 - w^2)(c^2 - w^2)]^{\frac{1}{2}}\, UU_1W'$$

$$\int \frac{V dv}{(v^2 - w^2)[(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}}}$$

$$+ A^2U^2 \int \frac{WW'\, dw}{(v^2 - w^2)} - AB\, [(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}}\, UU_1\, V'$$

$$\int \frac{W\, dw}{(v^2 - w^2)[(b^2 - w^2)(c^2 - w^2)]^{\frac{1}{2}}} \Bigg] \quad \ldots (7)$$

Here $U(u)$, $U_1(u)$, $V(v)$ and $W(w)$ are represented by U, U_1, V and W, respectively, and U', U_1', V', W' represent the differentials .

By chosing different suitable sets of values of U, U_1, V and W we may get a number of self-superposable fluid velocities . One of such velocities can be obtained by taking

$$\left.\begin{aligned} U_1 &= U = u \\ V &= [(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}} \\ W &= [(b^2 - w^2)(c^2 - w^2)]^{\frac{1}{2}} \\ A &= B \end{aligned}\right\} \quad \ldots (8)$$

the fluid velocity will become

$$\left.\begin{aligned} q_u &= 0 \\ q_v &= Au\left[\frac{(b^2-w^2)(c^2-w^2)}{(v^2-w^2)}\right]^{\frac{1}{2}} \\ q_w &= Au\left[\frac{(v^2-b^2)(c^2-v^2)}{(v^2-w^2)}\right]^{\frac{1}{2}} \end{aligned}\right\} . \qquad \ldots (9)$$

For this flow

$$\theta = A^2U^2\left[\frac{3}{2}\frac{\{(b^2-w^2)(c^2-w^2)+(v^2-b^2)(c^2-v^2)\}}{(v^2-w^2)}\right.$$

$$\left.+\ 2\{(b^2+c^2)-(v^2+w^2)\}\ \log(v-w)-(v^2+w^2)\right]. \qquad \cdots (10)$$

When U, U_1, V and W are constant quantities, then

$$\left.\begin{aligned} q_u &= 0 \\ q_v &= \frac{C_1}{[(v^2-w^2)]^{\frac{1}{2}}} \\ q_w &= \frac{D_1}{[(v^2-w^2)]^{\frac{1}{2}}} \end{aligned}\right\} . \qquad \ldots (11)$$

For this fluid velocity

$$\theta = \frac{K_1}{(v^2-w^2)}\ \log u . \qquad \ldots (12)$$

Cass II : When $q_v = 0$. In this case the self-superposable flows may be

(i)
$$\left.\begin{aligned} q_u &= \frac{A_1 V_1 W_1}{U^2} \\ q_v &= 0 \\ q_w &= \frac{B_1 U_2 V_2}{[(v^2 - w^2)]^{\frac{1}{2}}} \end{aligned}\right\} \quad \cdots (13)$$

and

$$\theta = \Bigg[\frac{1}{(v^2 - w^2)} \Big\{ B_1^2 V_1^2 \int U_2 U_2' \, du + B_1 V_2^2 \int \frac{U_2^2}{u} \, du$$

$$+ A_1 B_1 \left[\frac{(b^2 - w^2)(c^2 - w^2)}{(v^2 - w^2)} \right]^{\frac{1}{2}} V_1 V_2 W_1' \int \frac{U_2}{u^3} \, du \Big\}$$

$$+ \frac{A_1 W_1^2}{u^4} \int \frac{V_1 V_1' \, dv}{[(v^2 - w^2)]^{\frac{1}{2}}}$$

$$+ B_1^2 U_2^2 \int \frac{V_2 V_2'}{[(v^2 - w^2)]^{\frac{1}{2}}} \, dv + \frac{A_1 V_1^2}{u^4} \int \frac{W_1 W_1'}{[(v^2 - w^2)]^{\frac{1}{2}}} \, dw$$

$$- \frac{A_1 B_1 (U_2 + U_2') V_1 V_2}{u^2} \int \frac{W_1 \, dw}{[(b^2 - w^2)(c^2 - w^2]^{\frac{1}{2}}} \Bigg] \cdots (14)$$

(ii)
$$\left.\begin{aligned} q_u &= \frac{A_1 V}{u^2} [(b^2 - w^2)(c^2 - w^2)]^{\frac{1}{2}} \\ q_v &= 0 \\ q_w &= \frac{A_1 uv}{[(v^2 - w^2)]^{\frac{1}{2}}} \end{aligned}\right\} \quad \cdots (15)$$

$$\theta = A_1^2 \left[\frac{u^2 v^2}{(v^2 - w^2)} - \frac{v^2 w}{u^2} \left\{ u + 1 - \frac{u\,(2w^2 - b^2 - c^2)}{(v^2 - w^2)^{3/2}} \right\} \right.$$

$$+ \frac{(v^2 + w^2)^{\frac{1}{2}}}{u^4} \left\{ (b^2 - w^2)\,(c^2 - w^2) + \frac{2}{3} v^2 (v^2 - w^2) \right.$$

$$\left. \left. - (2v^2 - b^2 - c^2) + u^6 \right\} \right] \quad \dots (16)$$

(iii)

$$\left.\begin{aligned} q_u &= \frac{C_2}{u^2} \\ q_v &= 0 \\ q_w &= \frac{D_2}{(v^2 - w^2)^{\frac{1}{2}}} \end{aligned}\right\} \quad \dots (17)$$

Case III : When $q_w = 0$, some self - superposable flows may be

(i)

$$\left.\begin{aligned} q_u &= \frac{A_2 V_3 W_2}{u^2} \\ q_v &= \frac{B_2 U_3 W_3}{(v^2 - w^2)^{\frac{1}{2}}} \\ q_w &= 0 \end{aligned}\right\} \quad .. (18)$$

$$\theta = \left[\frac{B_2^2 W_3^2}{(v^2 - w^2)} \int \frac{U_3^2}{u} + \int \frac{B_2^2 W_3^2}{(v^2 - w^2)} \int U_3 U_3'\, du \right.$$

$$- A_2 B_2 \frac{[(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}}}{(v^2 - w^2)^{3/2}} V_3' W_2 W_3 \int \frac{U_3\, du}{u^3}$$

$$+ \frac{A_2^2 W_2^2}{u^4} \int \frac{V_3 V_3' \, dv}{(v^2 - w^2)^{\frac{1}{2}}}$$

$$- \frac{A_2 B_2 U_3' W_2 W_3}{u^2} \int \frac{V_3 \, dv}{[(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}}}$$

$$- \frac{A_2 B_2 U_3' W_2 W_3}{u} \int \frac{V_3 \, dv}{[(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}}}$$

$$+ \frac{A_2^2 V_3^2}{u^4} \int \frac{W_2 W_2' \, dw}{(v^2 - w^2)^{\frac{1}{2}}} + B_2^2 U_3^2 \int \left[\frac{W_3 W_3' \, dw}{(v^2 - w^2)} \right] \quad \ldots (19)$$

(ii)
$$\left.\begin{aligned} q_u &= \frac{A_2 w \, [(v^2 - b^2)(c^2 - v^2)]^{\frac{1}{2}}}{u^2} \\ q_v &= \frac{A_2 u}{(v^2 - w^2)^{\frac{1}{2}}} \\ q_w &= 0 \end{aligned}\right\} \quad \ldots (20)$$

$$\theta = A_2^2 \left[\left(\frac{u^2 w}{v_2 - w^2} \right) - \tfrac{1}{2} u^2 \log (v^2 - w^2) \right.$$

$$+ \frac{w^2 v}{u} \left\{ \frac{b^2 + c^2 - 2 v^2}{(v^2 - w^2)^{3/2}} - 1 \right\}$$

$$+ \frac{(v^2 - w^2)^{\frac{1}{2}}}{u^4} \left\{ (v^2 - b^2)(c^2 - v^2) \right.$$

$$\left.\left. + \frac{1}{3} w^4 (b^2 + c^2 - 2 v^2) \right\} \right] \quad \ldots (21)$$

(iii)

$$\left.\begin{aligned} q_u &= \frac{C_3}{u^2} \\ q_v &= \frac{D_3}{(v^2 - w^2)^{\frac{1}{2}}} \\ q_w &= 0 \end{aligned}\right\} \quad \dots (22)$$

Case IV : When $q_u = q_v = 0$. In this case the possible solution to the equation (5) may be

$$\left.\begin{aligned} q_u &= 0 \\ q_v &= 0 \\ q_w &= \frac{B_3 U_4 V_4}{(v^2 - w^2)^{\frac{1}{2}}} \end{aligned}\right\} \quad \dots (23)$$

Case V : When $q_v = q_w = 0$, then a self-superposable flow may be

$$\left.\begin{aligned} q_u &= \frac{A_3 V_5 W_4}{u^2} \\ q_v &= 0 \\ q_w &= 0 \end{aligned}\right\} \quad \dots (24)$$

Case VI : When $q_u = q_w = 0$. In this case a flow may be

$$\left.\begin{aligned} q_u &= 0 \\ q_v &= \frac{A_4 U_5 W_5}{(v^2 - w^2)^{\frac{1}{2}}} \\ q_w &= 0 \end{aligned}\right\} \quad \dots (25)$$

In all the above cases U_n, V_n, W_n ($n = 1, 2, 3, \dots$) are the integrable functions of u, v and w respectively and A_n, B_n, C_n, D_n ($n = 1, 2, 3 \dots$) are the constants which may be determined by boundary conditions .

3. SUPERPOSABLE FLUID MOTIONS

It has already been shown that the hydrodynamic flows given by equations (6) and (24) are self–superposable . By the method discussed in preceding pages it can easily be shown that if $\mathbf{q}_1$ aud $\mathbf{q}_2$ be the two flows (6) and (24) then $\mathbf{p}$ for $\mathbf{q}_1 + \mathbf{q}_2$ can also be represented by the gradient of a scalar quantity . Thus, according to Ballabh [3], $\mathbf{q}_1$ and $\mathbf{q}_2$ will be mutually superposable and a flow

$$\left.\begin{aligned} q_u &= \frac{AV(v)\,W(w)}{u^2} \\ q_v &= \frac{BU(u)\,W(w)}{(v^2 - w^2)^{\frac{1}{2}}} \\ q_w &= \frac{CU(u)\,V(v)}{(v^2 - w^2)^{\frac{1}{2}}} \end{aligned}\right\} \qquad \ldots (26)$$

is possible. The same flow can be determined by mutually superposing the flows (13), (25) and (18), (23) .

4. PRESSURE DISTRIBUTION

It will be interesting to note that θ is nothing but be rnoulli function [1] given by

$$\theta = \frac{q}{2g} + h + \frac{p}{g} \qquad \ldots (27)$$

where q, g, h and p denote velocity, acceleration due to gravity, height above some horizontal plane of reference and the pressure head .

It is a well known fact for an incompressible fluid the pressure head P is given by [5]

$$p = P / \rho_0 + \text{constant}, \qquad \ldots (28)$$

where p is the pressure distribution .

Also, if the motion of the fluid be steady and slow then value of h can be taken, without much loss of generality, as u for the flows (6),(9), (11); as v for the flows (13) (15), (17) and w for (18), (20), (22) .

Thus for the flow (9) pressure distribution can be determined as

$$P = \Big[K_0 u^2 (b^2 + c^2 - v^2 - w^2) + K_1 u^2 (v^2 + w^2) \log (v - w) + K_2 u^2 + K_3 u + K_4 \Big] \quad \ldots (29)$$

where K_0, K_1, K_2, K_3 and K_4 are constants .

Similarly, for flow (11), taking $C_1 = D_1$ we have

$$P = \left[\frac{(K_0' + K_1' \log u)}{(v^2 - w^2)} + K_2' u + K_3' \right] \quad \ldots (30)$$

Here K_0', K_1', K_2', and K_3' are constants.

Similarly, the pressure distribution for the flows (6) , (13), (15), (17), (18) , (20) and (22) can easily be determined .

5. VORTICITY OF THE FLOW

It was shown by Ballabh [3] that for a self-superposable flow, vorticity is constant along its stream lines . If $\mathbf{T}$ is a unit tangent along a stream line then

$$\mathbf{T} \times \mathbf{q} = 0 \quad \ldots (31)$$

By equations (17) and (31) it can readily be shown be that

$$\mathbf{T} = \left[\left(\frac{v^2 - w^2}{u^2 + v^2 - w^2} \right)^{1/2}, 0, \frac{u}{[u^2 + v^2 - w^2]^{\frac{1}{2}}} \right] \quad \ldots (32)$$

Hence the vorticity of the flow (17) is constant along the curves represented by equation (32) . Similarly the curves of constant vorticity can also be found for other flows .

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THE GENERALIZED PROLATE SPHEROIDAL WAVE FUNCTION AND ITS APPLICATIONS

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ABSTRACT

Here we employ the generalized prolate spheroidal wave function in obtaining the formal solution of the partial differential equation related to a problem of heat conduction in an anisotropic material. This type of problems occur mainly in wood technology, soil mechanics, and the mechanics of solids of fibrous structure. The solution of the problem when the arbitrary function $F(x)$ is given in terms of Srivastava–Panda's multivariable H–function has also been obtained. The results of this paper unify and extend a large number of results established by various earlier workers.

1. INTRODUCTION

The solution of the following differential equation

$$(1 - x^2)\frac{d^2 y}{dx^2} + [(\beta - \alpha) - (\alpha + \beta + 2)x]\frac{dy}{dx} + [\xi(s) + s^2x^2]\,y = 0 \qquad \ldots (1.1)$$

is the generalized prolate spheroidal wave function (see, e.g., Gupta [3], p. 107 eqn. (2. 11)) which has been denoted as

$$\phi_n^{\alpha,\beta}(s,x) + \sum_{p=0}^{\infty} R_{p,n}^{\alpha,\beta}(s)\,P_{p+n}^{(\alpha,\beta)}(x) \qquad \ldots (1.2)$$

provided that $s = 0, \xi(0) = (n+p)(\alpha+\beta+n+p+1), p \geqslant 0.$

Recently, Srivastava and Panda [7] have introduced and studied the H-function of several complex variables by means of the multiple Mellin-Barnes integral (see also Srivastava, Gupta and Goyal [6], p. 251)

$$H[z_1,\ldots,z_r] \equiv H^{O,\varepsilon:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:(B',D');\ldots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}]; \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\ldots,z_r\right)$$

$$= (2\pi\omega)^{-r}\int_{L_1}\cdots\int_{L_r} U_1(s_1)\ldots U_r(s_r)\,V(s_1,\ldots,s_r)$$

$$z_1^{s_1}\ldots z_r^{s_r}\,ds_1\ldots ds_r,\ \text{where } \omega=\sqrt{-1}, \qquad \ldots (1.3)$$

$$U_i(s_i) = \frac{\prod_{j=1}^{u^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}s_i]}{\prod_{j=1+u^{(i)}}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}s_i]}$$

$$\cdot\frac{\prod_{j=1}^{V^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}s_i]}{\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}s_i]},\quad i=1,\ldots,r$$

$$V(s_1, \ldots s_r) =$$

$$\frac{\prod_{j=1}^{\varepsilon} \Gamma\left[1 - a_j + \sum_{i=1}^{r} \theta_j^{(i)} s_i\right]}{\prod_{j=\varepsilon+1}^{A} \Gamma\left[a_j - \sum_{i=1}^{r} \theta_j^{(i)} s_i\right] \prod_{j=1}^{C} \Gamma\left[1 - c_j + \sum_{i=1}^{r} \Psi_j^{(i)} s_i\right]}$$

and, for convergence,

$$|\arg z_i| < \tfrac{1}{2} T_i \pi, \; i, \ldots, r \qquad \ldots (1.4)$$

where

$$T_i = -\sum_{j=\varepsilon+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \theta_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}} \theta_j^{(i)}$$

$$- \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}} \delta_j^{(i)}$$

$$> 0, \; i = 1, \ldots, r \qquad \ldots (1.5)$$

We require the following result due to Chaurasia and Gupta ([1], *p*. 82, eqn. (13)) in our investigation :

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma} \phi_n^{\alpha,\beta}(s, x)\, H\left[z_1 (1-x)^{h_1}\right.$$

$$\left.(1+x)^{k_1}, \ldots, z_r (1-x)^{h_r} (x+1)^{k_r}\right] dx$$

$$= \sum_{p=0}^{\infty} R_{p,n}^{\alpha,\beta}(s)\, 2^{\rho+\sigma+1}$$

$$\sum_{m=0}^{n+p} \frac{(-n-p)_m (\alpha+\beta+n+p+1)_m}{(\alpha+1)_m\, m!}$$

$$H_m^*\, [\, z_1\, 2^{h_1 + k_1}, \ldots, z_r\, 2^{h_r + k_r}\,] \qquad \ldots (1.6)$$

where

$$H_m^*\, [\, z_1, \ldots, z_r\,] \equiv H^{0,\, \varepsilon + 2 : (u', v'), \ldots, (u^{(r)}, v^{(r)})}_{A + 2,\, C + 1 : (B', D'), \ldots, (B^{(r)}, D^{(r)})}$$

$$\left(\begin{matrix} [-\sigma : k_1, \ldots, k_r], \\ [(c) : \Psi', \ldots, \Psi^{(r)}], \end{matrix} \right.$$

$$[-m-\rho : h_1, \ldots, h_r], [(a) : \theta', \ldots, \theta^{(r)}] : [(b') : \phi'] ; \ldots ;$$

$$[-1-m-\rho-\sigma : h_1 + k_1, \ldots, h_r + k_r] : [(d') : \delta'] ; \ldots ;$$

$$\left. \begin{matrix} [(b^{(r)}) : \phi^{(r)}] ; \\ [(d^{(r)}) : \delta^{(r)}] : \end{matrix} \; z_1 2^{h_1 + k_1}, \ldots, z^r\, 2^{h_r + k_r} \right) \qquad \ldots (1.7)$$

The integral (1. 6) is valid for

$$Re\, (\rho + \sum_{i=1}^{r} h_i\, d_j^{(i)} / \delta_j^{(i)}) > -1,$$

$$Re\, (\sigma + \sum_{i=1}^{r} k_i\, d_j^{(i)} / \delta_j^{(i)} > -1,$$

$h_i > 0$, $k_i > 0$ and, of course, $T_i > 0$, $|\arg z_i| < \frac{1}{2}\pi T_i$,

$Re\,(\alpha) > -1$, $Re\,(\beta) > -1$, $i = 1, \ldots, r$; $J = 1, \ldots, u^{(i)}$

The orthogonality property of the generalized prolate spheroidal wave function (Gupta [3], p, 107, eqn. (3.1)) is recalled in the following form :

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta}\, \phi_n^{\alpha,\beta}(s, x)\, \phi_t^{\alpha,\beta}(s, x)\, dx$$

$$= N_{n,t}^{\alpha,\beta}\, \delta_{nt} \qquad \ldots (1.8)$$

where

$$\overset{\alpha,\beta}{N}_{n,t} = \sum_{p=0}^{\infty} \left(\overset{\alpha,\beta}{K}_{p,n}(s) \right)^2 .$$

$$\frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+p+1\;\Gamma(n+\beta+p+1)}{(2n+2p+\alpha+\beta+1)\,\Gamma(n+p+1)\;\Gamma(n+p+\alpha+\beta+1)}$$

and δ_{nt} is the Kronecker delta .

2. FORMULATION AND SOLUTION OF THE PROBLEM

The partial differential equation related to a problem of heat conduction in an anisotropic material has been obtained by Saxena and Nageria [5] and is given below :

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x} \right] - \frac{pc\nu}{\lambda}\frac{\partial u}{\partial x} + \frac{Q(x)}{\lambda} = \frac{pc}{\lambda}\frac{\partial u}{\partial t} \quad ...(2.1)$$

With the law of conductivity $K = \lambda(1-x^2)$, $Q(x)$ is the intensity of a continuous source of heat situated inside this solid . Let the Initial temperature of the rod be given by

$$u(x,0) = F(x) \quad ... (2.2)$$

If we take

$$\nu = \frac{\alpha-\beta}{q} ,\; Q(x) = -(\alpha+\beta)\lambda x \frac{\partial u}{\partial x} - s^2x^2\lambda u,\; q = \frac{pc}{\lambda} ,$$

then the solution of (2.1) can be written in the form

$$u(x,t) = \sum_{l=0}^{\infty} A_l\, e^{-B_l t} \overset{\alpha,\beta}{\phi}_{l}(s,x) \quad ... (2\,3)$$

Substituting this into equation (2.1) , we have

$$B_l = \frac{(l+p)(l+p+\alpha+\beta+1)}{q} .$$

In order to find the value of A_l, we make use of the initial condition (2.2). This gives

$$F(x) = \sum_{l=0}^{\infty} A_l \, \phi_l^{\alpha,\beta}(s,x) \qquad \dots (2.4)$$

Multiplying both sides of (2.4) by

$$(1-x)^{\alpha}(1+x)^{\beta}\, \phi_l^{\alpha,\beta}(s,x)$$

and integrating with respect to x from -1 to 1, we obtain the result

$$A_l = \frac{G_l}{N_{l,l}^{\alpha,\beta}} \qquad \dots (2.5)$$

$$\text{where } G_l = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, \phi_l^{\alpha,\beta}(s,x)\, F(x)\, dx \qquad \dots (2.6)$$

This is because the generalized prolate spheroidal wave function possesses the orthogonality property (1.8).

Hence the solution of the problem may be expressed as

$$u(x,t) = \sum_{l=0}^{\infty} G_l \left(N_{l,l}^{\alpha,\beta-1} \exp\left[-\frac{(l+p)(l+p+\alpha+\beta+1)}{q} t \right] \phi_l^{\alpha\,\beta}(s,x) \right. \qquad \dots (2.7)$$

If we take $s = 0$ in (2.7), we get the. solution of the problem in terms of the Jacobi polynomials given earlier by Saxena and Nageria ([5], pp. 1–6). Again, setting $\alpha = \beta = 0$ and $s = 0$ in (2.7), we get the solution of the problem in terms of Legendre functions given by Churchill ([2], p. 224).

3. EXAMPLES

Let

$$F(x) = (1-x)^{\rho-\alpha}\ (1+x)^{\sigma-\beta}$$

$$H[\, z_1 (1-x)^{h_1}\ (1+x)^{k_1}, \ldots, z_r (1-x)^{h_r}\ (1+x)^{k_r}\,]$$

then by (2.4) we have formally that

$$(1-x)^{\rho-\alpha} (1+x)^{\sigma-\beta}$$

$$H[\, z_1 (1-x)^{h_1} (1+x)^{k_1}, \ldots, z_r (1-x)^{h_r}\ (1+x)^{k_r}\,]$$

$$= \sum_{l=0}^{\infty} A_l\, \phi_l^{\alpha,\beta} (s, x) \qquad \ldots (3.1)$$

Equation (3.1) is valid since $F(x)$ is continuous and of bounded variation in the open interval $(-1,1)$.

Now multiplying both sides of (3.1) by $(1-x)^{\alpha}(1+x)^{\beta}$

$\phi_n^{\alpha,\beta}(s,x)$, $\alpha > -1$, $\beta > -1$ and integrating with respect to x from

-1 to 1, change the order of integration and summation (which is permissible) on the right, we obtain

$$\int_{-1}^{1} (1-x)^{\rho} (1+x)^{\sigma}\ \phi_n^{\alpha,\beta}(s,x)$$

$$H[\, z_1 (1-x)^{h_1}\ (1+x)^{k_1}, \ldots, z_r (1-x)^{h_r} (1+x)^{k_r}\,]dx$$

$$= \sum_{l=0}^{\infty} A_l \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \phi_l^{\alpha,\beta}(s,x)\, \phi_n^{\alpha,\beta}(s,x)\, dx \quad \dots (3.2)$$

Using the orthogonality property of the generalized prolate spheroidal wave function (1.8) on the right-hand side and the result (1.6) on the left hand side of (3.2), we obtain

$$A_l = \sum_{p=0}^{\infty} R_{p,n}^{\alpha,\beta}(s)\, 2^{\rho+\sigma+1}$$

$$\sum_{m=0}^{n+p} \frac{(-n-p)_m (\alpha+\beta+n+p+1)_m}{(\alpha+1)_m\, m!} \left\{ N_{l,l}^{\alpha,\beta} \right\}^{-1}$$

$$H_m^{*} [z_1 2^{h_1+k_1}, \dots, z_r 2^{h_r+k_r}] \quad \dots (3.3)$$

where H_m^{*} function is defined by (1.7) on substituting the value of A_l from (3.3) in (2.3) and using the lemma ([4], p.57, eqn (2)), we arrive at the solution of the problem (2.3) as

$$u(x,t) = \sum_{p=0}^{\infty} \sum_{l=0}^{p} \sum_{m=0}^{p} 2^{\rho+\sigma+1}$$

$$R_{p-l,l}^{\alpha,\beta}(s) \frac{(-p)_m (\alpha+\beta+p+1)_m}{(\alpha+1)_m\, m!} \left\{ N_{l,l}^{\alpha,\beta} \right\}^{-1}$$

$$H_m^{*} [z_1 2^{h_1+k_1}, \dots, z_r 2^{h_r+k_r}]\, e^{-(p/q)(p+\alpha+\beta+1)t}$$

$$\phi_l^{\alpha,\beta}(s,x) \quad \dots (3.4)$$

valid under the condition mentioned with (1 6) .

We notice that the multivariable H-function involved in our result (3 4) is quite general in character; Indeed, by suitably specializing the parameters of this function, we can easily obtain solution of the problem in terms of a large number of elementary functions of one or more variables (or product of several such functions).

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ON A GENERALIZED MULTIPLE TRANSFORM. I

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ABSTRACT

In the present paper we study a general multiple integral transform in which the kernel is the product of Fox's H- functions and the H- function of several variables which was introduced by H. M. Srivastava and R. Panda Our transform provides interesting extensions of various classes of known integral transforms of many variables whose kernels are expressible in terms of E, G and H- functions of one and more variables. Several special cases of this transform have been discussed briefly. we lastly give certain elementary and useful properties for our transform.

1. DEFINITION

we study the multiple H– function tranform defined by

$$\phi(s_1 ,\cdots, s_r) = H_r \{ f : s_1,\ldots,s_r \} \tag{1}$$

$$= (s_1 ,\ldots, s_r) \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ H^{u_i , 0}_{v_i , w_i} \left[t_i s_i x_i \left| \begin{matrix} (k_j^{(i)} , \rho_j^{(i)})_{1 , v_i} \\ (h_j^{(i)} , \sigma_j^{(i)})_{1 , w_i} \end{matrix} \right. \right] \right\} H^{0 , 0 : m_1 , n_1 ;\ldots; m_r , n_r}_{p , q : p_1 , q_1 ;\ldots; p_r , q_r} \left[\begin{matrix} \lambda_1 s_1 x_1 \\ \vdots \\ \lambda_r s_r x_r \end{matrix} \right|$$

$$\left.\begin{matrix} (a_j\ ;\ \alpha_j',\ldots,\alpha_j^{(r)})_{1,p} : \\ (b_j\ ;\ \beta_j',\ldots,\beta_j^{(r)})_{1,q} : \\ (c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(n)})_{1,p_r} \\ (d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{matrix}\right] f(x_1,\ldots,x_r)\,dx_1\ldots dx_r.$$

The H-function of one variable appearing in (1.1) was first introduced by Fox [3]. The details of this function can be found in a recent monograph by Srivastava, Gupta and Goyal [8]. On the other hand, the multivariable H-function :

$$(1.2)\quad H^{0,0:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (a_j\ ;\ \alpha_j',\ldots,\alpha_j^{(r)})_{1,p} : \\ (b_j\ ;\ \beta_j',\ldots,\beta_j^{(r)})_{1,q} : \end{matrix}\right.$$

$$\left.\begin{matrix} (c_j',\gamma_j')_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (d_j',\delta_j')_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)},)_{1,q_r} \end{matrix}\right] = H_1[z_1,\ldots,z_r]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\ldots,s_r)\,\theta_1(s_1)\ldots\theta_r(s_r)\,z_1^{s_1}\ldots z_r^{s_r}$$

$$.\,ds_1\ldots ds_r,\ \omega=\sqrt{-1},$$

is a *particular* case of the H-function of several complex variables (or the multivariable H-function) which was introduced and studied by Srivastava and Panda in a series of papers ([5], [6] and [7]; See also [8], p. 251, Eq. (C.1)).

The multiple integral on the right-hand side of (1.1) exists and defines the multiple H-function transformation of a given function $f(x_1,\ldots,x_r)$ under the following *sufficient* conditions :

(i) $T_i = \sum_{j=1}^{w_i} \sigma_j^{(i)} - \sum_{j=1}^{v_i} \rho_j^{(i)} < 0$;

(ii) $S_i = \sum_{j=1}^{u_i} \sigma_j^{(i)} - \sum_{j=u_i+1}^{w_i} \sigma_j^{(i)} - \sum_{j=1}^{v_i} \rho_j^{(i)} > 0$,

$| \arg t_i s_i | < (1/2) S_i \pi$;

(iii) $\Lambda_i > 0$, $| \arg \lambda_i s_i | < (1/2) \Lambda_i \pi$, where

$$\Lambda_i = - \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{n_i} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} \gamma_j^{(i)}$$

$$- \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{m_i} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} \delta_j^{(i)} > 0 ,$$

$$\forall\ i \in \{ 1 ,..., r \} ;$$

(iv) $f(x_1, ..., x_r)$ is a real-or complex-valued function of r variables $x_1, .., x_r$ defined on $R : 0 \leqslant x_1 < \infty, ..., 0 \leqslant x_r < \infty$ and such that the product

$\prod_{i=1}^{r} \left\{ x_i^{\xi_i + n_i} \right\} f(x_1, ..., x_r)$ is integrable (in the sense of

Lebesgue) over $R(R_1, R_2, ..., R_n) : 0 \leqslant x_i \leqslant R_i$; $R_j > 0$,

where

$$\xi_i = \min_{1 \leqslant j \leqslant u_i} \left\{ Re \left(h_j^{(i)} / \sigma_j^{(i)} \right) \right\} \text{ and}$$

$$\eta_i = \min_{1 \leqslant j \leqslant m_i} \left\{ Re \left(d_j^{(i)} / \delta_j^{(i)} \right) \right\} \forall\ i \in \{ 1, ..., r \} ,$$

and

(v) the limit of the finite form of the multiple integral in (1.1) with

$\int_0^\infty \cdots \int_0^\infty$ replaced by $\int_0^{R_1} \cdots \int_0^{R_r}$ exists at the point ($s_1, \ldots, s_r$)

when $R_1, \ldots, R_r \to \infty$.

2. SPECIAL CASES OF THE INTEGRAL TRANSFORM DEFINED BY (1.1)

The integral transform (1.1) is quite general in character ; indeed its various specialized and confluent forms can be suitably related to the several classes of known integral transforms of one, two and more variables considered in the literature from time to time by various research workers . Some interesting connections † of this transform with the known integral transforms are mentioned below :

If we take $u_i = w_i = 1$, $v_i = 0$, $\sigma_1^{(i)} = 1$, $h_1^{(i)} = 0$ in (1.1) and using a known result [8, p. 18, *Eq* . (2.6.2)], we get

$$(2.1)\quad \phi (s_1, \ldots, s_r) = (s_1 \ldots s_r) \int_0^\infty \cdots \int_0^\infty exp \left\{ - \sum_{i=1}^{r} (s_i\, t_i\, x_i) \right\}$$

$$H_1 [\lambda_1 s_1 x_1, \ldots, \lambda_r s_r x_r] f (x_1, \ldots, x_r) \, dx_1 \ldots dx_r$$

provided that the multiple integral involved in (2.1) is absolutely convergent .

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† Similar multiple *H*-function transformations have been studied by Y. N. Prasad. and K. Nath [*Istanbul Tek. Üniv . Bul* . **36** (1983), 361–371 ; *Vijnana Parishad Anusandhan Patrika* **27** (1984), 23–43] .

The integral transform (2 1) is clearly related to the transform studied earlier by Srivastava and Panda [7, *p.* 119, *Eq* . (1.1)] . Further, if we let $t_i \to 0$ ($i = 1, \ldots, r$) , the integral transform (2.1) will reduce to another integral transform studied by Srivastava and Panda [7, p. 121, *Eq* . (1.15)] .

$$(2.2)\quad \phi\,(s_1, \ldots, s_r) = (s_1 \ldots s_r) \int_0^\infty \cdots \int_0^\infty H_1 [\lambda_1 s_1 x_1, \ldots, \lambda_r s_r x_r] \; . f (x_1, \ldots, x_r)\, dx_1 \ldots dx_r \,,$$

provided that the multiple integral in (2.2) converges absolutely .

Again, if we take $r = 2$ in (1.1), we shall obtain the following double integral transform introduced and studied by Agarwal [1, *p.* 131] :

$$(2.3)\quad \phi\,(s_1, s_2) = s_1 s_2 \int_0^\infty \int_0^\infty H^{u_1, 0}_{v_1, w_1} \left[t_1 s_1 x_1 \,\middle|\, \begin{matrix} (k_j', \rho_j')_{1, v_1} \\ (h_j', \sigma_j')_{1, w_1} \end{matrix} \right]$$

$$. H^{u_2, 0}_{v_2, w_2} \left[t_2 s_2 x_2 \,\middle|\, \begin{matrix} k_j'' : \rho_j'')_{1, v_2} \\ h_j'' , \sigma_j'')_{1, w_2} \end{matrix} \right]$$

$$H_1 [\lambda_1 s_1 x_1, \lambda_2 s_2 x_2] f (x_1, x_2) \; dx_1 \, dx_2 \,,$$

provided that double integral (2.3) converges absolutely .

In (2.3), $H_1 [x,y]$ stands for the H-function of two variables (with $n = 0$) studied earlier by Mittal and Gupta ([4] ; see also [8]).

We should mention that the aforementioned transforms (2.1) and (2.3) are quite general in nature and include a large number of other well-known integral transforms of one, two and several variables . For details , refer to the works by Srivastava and Panda [7], and also by Agrawal [2] .

3. ELEMENTARY PROPERTIES

Property I.

$$(3.1)\quad \int_0^\infty \cdots \int_0^\infty g(s_1, \ldots, s_r)\, H_r\left\{ f : s_1, \ldots, s_r \right\} \frac{ds_1}{s_1} \cdots \frac{ds_r}{s_r}$$

$$= \int_0^\infty \cdots \int_0^\infty f(s_1, \ldots, s_r)\, H_r\left\{ g : s_1, \ldots, s_r \right\} \frac{ds_1}{s_1} \cdots \frac{ds_r}{s_r}$$

provided the multiple integrals converge absolutely.

Property II.

If the multidimensional H_r-integral transform of $f(x_1, \ldots, x_r)$ defined by (1.1) exists, then

$$(3.2)\quad H_r\left\{ f(\mu_1 x_1, \ldots, \mu_r x_r) : s_1, \ldots, s_r \right\}$$

$$= H_r\left\{ f(x_1, \ldots, x_r) : \frac{s_1}{\mu_1}, \ldots, \frac{s_r}{\mu_r} \right\}$$

provided that $\mu_1, \ldots, \mu_r > 0$.

Evidently, this property may be called similarity rule for the multiple H_r-transform defined by (1.1)

Property III.

If the multidimensional H_r-transform of $f(x_1, \ldots, x_r)$ defined by (1.1) exists, then

$$(3.3)\quad \sum_{j=1}^{N} \triangle_j H_r\left\{ f_j : s_1, \ldots, s_r \right\} = \sum_{j=1}^{N} H_r\left\{ \triangle_j f_j : s_1, \ldots, s_r \right\}.$$

where $\triangle_1, \ldots, \triangle_N$ are constants. This property may be looked upon as linearity rule for the multiple H_r-integral transform.

In our next communication we shall give an inversion formula and study some other important properties of the transform (1.1).

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AN EXPANSION FORMULA FOR THE H-FUNCTION OF SEVERAL VARIABLES

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ABSTRACT

The aim of the present paper is to establish a general expansion formula for H-function of several variables which was introduced and studied in a series of papers by H. M. Srivastava and R. Panda (see, for example, [4] and [5]). It is significant to observe that a large number of finite and infinite series for this function can be easily summed up by using the well-known summation theorems for ordinary hypergeometric series in the main result. Also, by appropriately specializing the parameters of the H-function of several variables, one can easily obtain expansion formulas for simpler special functions of one and more variables.

1. INTRODUCTION

We begin by recalling the definition of the H-function of several variables introduced and studied by Srivastava and Panda ([4] and [5] ; see also Srivastava. Gupta and Goyal [3 .p. 251, Eq . (C.1)]) :

$$H[z_1,\ldots,z_r] = H^{0,\, n\,:\, m_1,\, n_1\,;\ldots;\, m_r,\, n_r}_{p,\, q\,:\, p_1,\, q_1\,;\ldots;\, p_r,\, q_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\right|$$

$$\left.\begin{matrix} (a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : (c_j', \gamma_j')_{1,p_1} ; \ldots ; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,q} : (d_j', \delta_j')_{1,q_1} ; \ldots ; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L^r} \phi_1(\xi_1) \ldots \phi_r(\xi_r) \Psi(\xi_1, \ldots, \xi_r) \cdot z_1^{\xi_1} \ldots z_r^{\xi_r} \, d\xi_1 \quad d\xi^r \tag{1.1}$$

where $w = \sqrt{-1}$,

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)}$$

$$i \in \{1, \ldots, r\} \tag{1 2}$$

and

$$\Psi(\xi_1, \ldots, \xi_r) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{j=1}^{r} \alpha_j^{(i)} \xi_i)}{\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i) \prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i)} . \tag{1.3}$$

The nature of the contours in (1.1), the asymptotic expansions, conditions of convergence of (1.1) and particular cases of the multivariable H-function can be found in the papers [4] and [5]. See also the book by Srivastava, Gupta and Goyal [3, p 251].

2. MAIN RESULT

$$\sum_{k=0}^{\infty} \frac{\prod\limits_{j=1}^{A} (a_j')_k \, z^k}{\prod\limits_{j=1}^{B} (b_j')_k \, k\,!} \; H^{0,\, n+C \;:}_{p+C+D,\, q+E\;:}$$

$$m_1 + F_1, n_1 + G_1 \quad ;\dots ;m_r + F_r, n_r + G_r$$

$$p_1 + G_1 + H_1, q_1 + F_1 + I_1 ; \dots ;p_r + G_r + H_r, q_r + F_r + I_r$$

$$\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{l} (u_j - k; \rho_j' ;\dots, \rho_j^{(r)})_{1,C}, (a_j ; \alpha_j',\dots, \alpha_j^{(r)})_{1,p}, \\ (b_j ; \beta_j',\dots, \beta_j^{(r)})_{1,q}, (w_j - k; \epsilon_j',\dots, \epsilon_j^{(r)})_{1,E} \end{array}\right.$$

$$(v_j + k ; \sigma_j',\dots, \sigma_j^{(r)})_{1,D} : (g_j - k, \eta_j)_{1,G_1},$$

$$: (i_j' + k, \mu_j')_{1,F_1}, (d_j', \delta_j')_{1,q_1}, (l_j' - k, u_j')_{1,I_1} ;$$

$$(c_j', \gamma_j')_{1,p_1}, (h_j' + k, \xi_j')_{1,H_1} ;$$

$$\dots ; (i_j^{(r)} + k, \mu_j^{(r)})_{1,F_r},$$

$$\left.\begin{array}{l} \dots ; (g_j^{(r)} - k, \eta_j^{(r)})_{1,G_r}, (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r}, (h_j^{(r)} + k, \xi_j^{(r)})_{1,H_r} \\ \qquad (d_j^{(r)}, \delta_j^{(r)})_{1,q_r}, (l_j^{(r)} - k, v_j^{(r)})_{1,I_r} \end{array}\right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi_1(\xi_1) \dots \phi_r(\xi_r)\, \Psi(\xi_1 \dots \xi_r)$$

$$\Gamma\left[\begin{array}{c} A_1(\xi_1,\dots,\xi_r) \\ B_1(\xi_1,\dots,\xi_r) \end{array}\right] {}_{A+P}F_{B+Q}\left[\begin{array}{c} (a_A), A_1(\xi_1,\dots,\xi_r) \\ (b_B'), B_1(\xi_1,\dots,\xi_r) \end{array}\right]$$

$$\cdot\, z_1^{\xi_1} \dots z_r^{\xi_r} \, d\xi_1 \dots d\xi_r, \tag{2.1}$$

where

$A_1 (\xi_1 ,.., \xi_r)$ stands for

$$(1 - u_j + \sum_{i=1}^{r} \rho_j^{(i)} \xi_i)_{1,C}, \; (1 - g_j' + \eta_j' \xi_1)_{1,G_1} ,...,$$

$$(1 - g_j^{(r)} + \eta_j^{(r)} \xi_r)_{1,G_r} , (i_j' - \mu_j' \xi_1)_{1,F_1} , \ldots , (i_j^{(r)} - \mu_j^{(r)} \xi_r)_{1,F_r}$$

$B_1 (\xi_1 ,..., \xi_r)$ stands for

$$(v_j - \sum_{i=1}^{r} \sigma_j^{(i)} \xi_i)_{1,D} , (1 - w_j + \sum_{i=1}^{r} \varepsilon_j^{(i)} \xi_i)_{1,E} ,$$

$$(h_j' - \xi_j' \xi_1)_{1,H_1} ,\ldots, (h_j^{(r)} - \xi_j^{(r)} \xi_r)_{1,H_r} ,$$

$$(1 - l_j' + v_j' \xi_1)_{1,I_1} ,..., (1 - l_j^{(r)} + v_j^{(r)} \xi_r)_{1,I_r} .$$

Formula (2.1) holds if any one of the following sets of conditions is satisfied

(i) $A + P < B + Q + 1$

(ii) $A + P = B + Q + 1, \; | z | < 1$

(iii) $A + P = B + Q + 1, \; Re (\theta) > 0 , \; z = 1$

(iv) $A + P = B + Q + 1, \; Re (\theta + 1) > 0 , \; z = - 1$

where

$$P = C + \sum_{i=1}^{r} (G_i + F_i)$$

$$Q = D + E + \sum_{i=1}^{r} (H_i + I_i)$$

$$\theta = \sum_{j=1}^{B} b_j' - \sum_{j=1}^{A} a_j' - \left\{ \sum_{i=1}^{r} \left(\sum_{j=1}^{F} i_j^{(r)} \right.\right.$$

$$+ \sum_{j=1}^{I_i} l_j^{(r)} - \sum_{j=1}^{G_i} g_j^{(r)}$$

$$\left.\left. - \sum_{j=1}^{H_i} h_j^{(r)} \right) + \sum_{j=1}^{E} w_j - \sum_{j=1}^{C} u_j - \sum_{j=1}^{D} v_j \right\}$$

$$+ E - C + \sum_{i=1}^{r} (I_i - G_i)$$

Derivation of (2.1) . To evaluate (2.1) , we first replace H-function of several variables, occurring on the left-hand side of (2.1) by the contour integral in (1.1) , and change the order of integration and summation [which is justified under the conditions mentioned with (1.1)] ; then we get the required result (2.1) on using the definition of the generalized hypergeometric function (see, e g., [2] , p. 40; [3] , p. 2) . The routine details may be omitted .

3. PARTICULAR CASES

(i) If in (2.1) $A = 2, B = 0, P = 0, Q = 1$ ($E = 1$) $z = 1$, using the Gauss theorem [2, p 243 (III. 3)] , we get the following summation formula :

$$\sum_{k=0}^{\infty} \frac{(a_1')_k (a_2')_k}{k!}$$

$$H \left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{l} (a_j ; \alpha_j', \dots, \alpha_j^{(r)})_{1,p} \\ (w_1 - k ; \varepsilon_1', \dots, \varepsilon_1^{(r)}), (b_j ; \beta_j', \dots, \beta_j^{(r)})_{1,q} \end{array} \right.$$

$$\left. \begin{matrix} : (c_j', \gamma_j')_{1,p_1} ;\ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ : (d_j', \delta_j')_{1,q_1} ;\ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$= H \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (w_1 + a_1' + a_2' ; \varepsilon_1', \ldots, \varepsilon_1^{(r)}), (a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : \\ (w_1 + a_1' ; \varepsilon_1', \ldots, \varepsilon_1^{(r)}) (w_1 + a_2' ; \varepsilon_1', \ldots, \varepsilon_1^{(r)}), \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j', \gamma_j')_{1,p_1} ;\ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,q} : (d_j', \delta_j')_{1,q_1} ;\ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right] \tag{3.1}$$

(ii) If we put $A = 1, B = 0, P = Q = 1\ (E = C = 1)$, $z = -1$ in (2.1) and use Kummer's theorem [2, *p.* 243 (III.5)], we get the following result :

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (u_1 - k ; \rho_1', \ldots, \rho_1^{(r)}), \\ (-1 + a_1' + u_1 - k ; \rho_1', \ldots, \rho_1^{(r)}), \end{matrix} \right.$$

$$\left. \begin{matrix} (a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : (c_j', \gamma_j')_{1,p_1} ;\ldots; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r} \\ (b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,q} : (d_j', \delta_j')_{1,q_1} ;\ldots; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix} \right]$$

$$= H \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (u_1 ; \rho_1', \ldots, \rho_1^{(r)}), \left(-\frac{1}{2} + \frac{u_1}{2} ; \frac{\rho_1'}{2}, \ldots, \frac{\rho_1^{(r)}}{2}\right), \\ (-1 + u_1 ; \rho_1', \ldots, \rho_1^{(r)}), \end{matrix} \right.$$

$$(a_j ; \alpha_j', \ldots, \alpha_j^{(r)}) : (c_j', \gamma_j)_{1,p_1}$$

$$\left(-\frac{1}{2} + \frac{u_1}{2} + a_1' ; \frac{\rho_1'}{2}, \ldots, \frac{\rho_1^{(r)}}{2}\right), (b_j ; \beta_j' ;\ldots; \beta_j^{(r)}) : d_j', \delta_j')_{1,q_1}$$

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$$;\ldots;\ c_j^{(r)},\ \gamma_j^{(r)}\ )_{1,\,p_r}\\ ;\ldots;\ (\ d_j^{(r)}\ ,\ \delta_j^{(r)}\ )_{1,q_r}\Bigg] \tag{3.2}$$

It is thus seen that a large number of summation formulas can be obtained from our main result (2.1). Hence our main result may be regarded as the key formula for the summation of series involving the H-function of several variables. It is not out of place to mention that the results obtained by Vashishta and Goyal [6], Verma [7], Parihar [1], and others, can be easily obtained as special cases of main result (2.1),

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SOME THEOREMS FOR A DISTRIBUTIONAL GENERALIZED LAPLACE TRANSFORM

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ABSTRACT

A generalization of the Laplace transform was qiven by professor H. M. Srivastava in the year 1968. For this general integral transform S. K. Sinha (1981) gave complex inversion and Tauberian theorems, and A. K. Tiwari and A. Ko (1982) gave several further properties in the distributional sense including complex inversion and Abelian theorems . We extend this generalization to distributions and obtain similar Abelian theorems of the initial and final value types [see, also the recent work of V. G. Joshi and R.K. Saxena (1981) involving an analogous study of a substantially more general *H*–transform] .

1. Introduction and Definitions

A generalization of the Laplace transform

$$L[\,f(t) : p\,] = \int_0^\infty e^{-pt}\, f(t)\, dt\ ,\ Re\, p > 0\,. \tag{1}$$

was given by Srivastava [5] in the form (see also [6]) :

$$S^{(\rho,\sigma)}_{q,k,m}\,[f(t):p] =$$

$$\int_0^\infty (pt)^{\sigma-1/2}\; e^{-qpt/2}\; W_{k,m}\,(\rho pt) f(t)\,dt \tag{2}$$

where $W_{k,m}(z)$ denotes the whittaker function (Whittaker and Watson [8, p 339]) . By analogy with Srivastava's definition (2), we define the generalized Laplace transform of a distribution $f(t)$, whose support is bounded on the left , by (see also [3] and [7])

$$F(p) = S^{(\rho,\sigma)}_{q,k,m}\,[f(t):p]$$

$$= \langle f(t),\; e^{-qpt/2}\; (pt)^{\sigma-1/2}\; W_{k,m}\;(\rho pt)\rangle \tag{3}$$

In this paper we prove Abelian theorems of the initial and final value types for the distributional (Srivastava's) generalized Laplace transform (3) . For an analogous study of a substantially more general H-transform, see a recent work [2] .

Let us assume that there exists a real number c for which $e^{-ct} f(t)$ is a distribution of slow growth . We may convert (3) into

$$F(p) = S^{(\rho,\sigma)}_{q,k,m}\,[f(t): p]$$

$$= \langle e^{-ct}\, f(t),\; \lambda(t)\, e^{-(qp-2c)\,t/2}\; (pt)^{\sigma-1/2}\; W_{k,m}\;(\rho pt)\rangle \tag{4}$$

where $\lambda(t)$ is any infinitely smooth function (with support bounded on the left) which equals one over a neighbourhood of support of $f(t)$.

The definition (4) possesses a sense since

$$\left\{ \lambda(t) \quad e^{-(qp - 2c)\, t/2} \, (pt)^{\sigma - 1/2} \, W_{k,m} \, (\rho p t) \right\}$$

is a testing function in S_t , which denotes the space of testing functions of rapid descent .

2. Abelian Theorems of the Initial Value Type

We relate here the asymptotic behaviour of $f(t)$ as $t \to 0+$ to the asymptotic behaviour of $F(p)$ as $p \to \infty$. Let us assume that $f(t)$ is a locally integrable function and satisfies Condition A, that is,

(i) $f(t) = 0$ for $-\infty < t < T$, and

(ii) there exists a real number c such that $\left\{ e^{-ct} \quad f(t) \right\}$

is absolutely integrable over $-\infty < t < \infty$.

Theorem 1 (cf . [7, p. 352, Theorem 4.1]) . *If the locally integrable function $f(t)$ satisfies Condition A with $T = 0$ (that is, the support of $f(t)$ is in $0 \leqslant t < \infty$), and if there exist a complex number α and a real number η ($\eta > -1$) such that*

$$\lim_{t \to 0+} \left\{ \frac{f(t)}{t^{\eta}} \right\} = \alpha, \tag{5}$$

then

$$\lim_{p \to \infty} \left\{ \frac{p^{\eta+1} \, F(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \right\} = \alpha \tag{6}$$

where

$$\beta(\eta, m, k, \rho, \sigma, q) =$$

$$\frac{\Gamma(\eta + m + \sigma + 1)\ \Gamma(\eta + \sigma - m + 1)\ \rho^{m+1/2}}{\Gamma(\eta + \sigma - k + 3/2)\,[\tfrac{1}{2}(q + \rho)]^{m + \eta + \sigma + 1}}$$

$$\cdot\ {}_2F_1\left(\eta + m + \sigma + 1,\ m - k + 1/2,\ \eta + \sigma - k + 3/2, \frac{q - \rho}{q + \rho}\right)$$

and

$$F(p) = \int_0^\infty e^{-qpt/2} (pt)^{\sigma - 1/2}\, W_{k,m}(\rho pt) f(t)\, dt \qquad (7)$$

Proof. We first note that for $\eta > -1$ and $p > 0$.

$$\int_0^\infty t^\eta e^{-qpt/2} (pt)^{\sigma - 1/2}\, W_{k,m}(\rho pt)\, dt$$

$$= \frac{1}{p^{\eta+1}}\, \beta(\eta, m, k, \rho, \sigma, q) \qquad (8)$$

provided that $\eta + \sigma \pm m + 1 > 0$ and $q + \rho > 0$ (see Erdélyi et al. [1, p 216 Eq. (16)]). By using this result and assuming that $p > 0$ and $y > 0$. we may write

$$|p^{\eta + 1} F(p) - \alpha\, \beta(\eta, m, k, \rho, \sigma, q)|$$

$$= |p^{\eta + 1} \int_0^\infty e^{-qpt/2} (pt)^{\sigma - 1/2}\, W_{k,m}(\rho pt) \{f(t) - \alpha t^n\} dt|$$

$$\leqslant |p^{\eta + 1} \int_0^y e^{-qpt/2} (pt)^{\sigma - 1/2}\, W_{k,m}(\rho pt) \{f(t) - \alpha t^\eta\}\, dt|$$

$$+ \mid p^{\eta + 1} \int_y^{\infty} e^{-qpt/2} (pt)^{\sigma - 1/2} W_{k,m} (\rho pt) \{ f(t) - \alpha t^n \} dt \mid$$

$$= I_1 + I_2 \tag{9}$$

Let us first consider I_1 of (9)

$$I_1 = \mid p^{\eta + 1} \int_0^y t^{\eta} e^{-qpt/2} (pt)^{\sigma - 1/2} W_{k,m} (\rho pt)$$

$$\left\{ \frac{f(t)}{t^n} - \alpha \right\} dt \mid$$

$$\leqslant \sup_{0 \leqslant t \leqslant y} \left| \frac{f(t)}{t^n} - \alpha \right| . \beta (\eta, m, k, \rho, \sigma, q) \tag{10}$$

Now for I_2, we note from the second part of Condittion A and the assumption $c > 0$, that $e^{-ct} \mid f(t) - \alpha t^n \mid$ is absolutely integrable over $0 < t < \infty$. So, for $p > \dfrac{2c}{q}$.

$$I_2 = \mid p^{\eta + 1} \int_y^{\infty} \left\{ e^{-qpt/2} (pt)^{\sigma - 1/2} W_{k,m} (\rho pt) e^{ct} \right\}.$$

$$e^{-ct} \left\{ f(t) - \alpha t^n \right\} dt \mid$$

Now since the function

$$e^{-(qp - 2c) t/2} (pt)^{\sigma - 1/2} W_{k,m} (\rho pt)$$

is bounded in (Y, ∞) , so it must have an upper bound . Let the upper bound of the function be attained at the point $t = \xi$. Then

$$I_2 \leqslant p^{\eta+1} \mid e^{-(qp-2c)\xi/2} (p\xi)^{\sigma-1/2} W_{k,m}(\rho p\xi) \mid .$$

$$\int_y^\infty e^{-ct} \mid f(t) - \alpha t^\eta \mid dt$$

$$\leqslant KP^{\eta+1} e^{-(qp-2c)\xi/2} (p\xi)^{\sigma-1/2} W_{k,m}(\rho p\xi) \qquad (11)$$

where $\qquad K = \int_y^\infty e^{-ct} \mid f(t) - \alpha t^\eta \mid dt$.

Now let $\varepsilon > 0$. From relation (5) it is clear that y can be chosen so small that the right-hand side of (10), which is independent of p, becomes less than ε. By fixing y in this way, we can choose p very large so that the right-hand side of (11) becomes less than ε. Since ε is arbitrary, combining the results of (10) and (11), we find that the left hand side of (6) can be made arbitrarily small for large p. This proves (6). Thus

$$\mid p^{\eta+1} F(p) - \alpha\beta(\eta, m, k, \rho, \sigma, q) \mid \leqslant \varepsilon \text{ for } p > \frac{2c}{q}$$

or

$$\frac{P^{\eta+1} F(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \to \alpha \quad \text{as } p \to \infty .$$

which proves the theorem .

Now weextend this theorem to a distribution which, in some neighbourhood of the origin, corresponds to an ordinary function . In order to make this extension , we shall need the following

Lemma. *Let $f(t)$ be a generalized Laplace transformable distrtbution with support in $y \leqslant t < \infty$, where $y > 0$. Let the real number c be such that $e^{-ct} f(t)$ is in S', which denotes the space of all distributions of slow growth. Then, for $(1+2c)/q < p < \infty$,*

$$| F(p) | \leqslant M\, e^{-qp\tau/2}$$

where M is a constant and τ is any real number satsfying $0 < \tau < y$.

Proof. For $p > 2c/q$,

$$F(p) = < e^{-ct} f(t) , \lambda(t)\; e^{-(qp-2c)t/2} (pt)^{\sigma-1/2}\; W_{k,m}(\rho pt)>$$

where $\lambda(t)$ is an infinitely smooth function whose support in $x \leqslant t < \infty$. Here x is a fixed number satisfying $0 < x < y$. Again distribution of slow growth satisfies a boundedness condition of the type

$$| < f , \phi > | \leqslant c \operatorname*{Sup}_{t} | (1 + t^2)^r \phi^{(r)}(t) |$$

for given f in S and for ϕ in S, where the constant c and the nonnegative integer r depend only on f.

Thus

$$| F(p) | = | < e^{-ct} f(t), \lambda(t) | e^{-(qp-2c)t/2} (pt)^{\sigma-1/2} W_{k,m}(\rho pt)> |$$

$$\leqslant c \operatorname*{Sup}_{x < t < \infty} | (1 + t^2)^n \frac{d^n}{dt^n} [\lambda(t)\, e^{-(qp-2c)t/2} (pt)^{\sigma-1/2} . W_{k,m}(\rho pt)] |$$

$$= c \operatorname*{Sup}_{x \leqslant t < \infty} | (1 + t^2)^n \sum_{r=0}^{n} \binom{n}{r} \lambda(t)^{(n-r)} = \sum_{v=0}^{r} \binom{r}{v} (-1)^{r-v}.$$

$$\left\{\frac{1}{2} - (q-p)p - c\right\}^{r-\nu} e^{-(qp-2c)t/2} \rho^{-m+1/2}$$

$$\sum_{u=0}^{\nu} \binom{\nu}{u} \frac{(\sigma-m)!}{(\sigma-m-\nu+u)!} \frac{(-1)^u (pt)^{\sigma-\nu+(u-1)/2}}{p^{\nu} \rho^{m+(u-1)/2}}$$

$$. \; W_{k+u/2,\, m-u/2}(\rho pt) \,|\,,\ \sigma \geqslant m+\nu \text{ (see Slater [4, } p\text{ . 25]) .} \qquad (12)$$

where the constant c and the integer $n < m$ depend only on f. Let us consider

$$\phi(\rho pt) = (pt)^{\sigma-\nu+(u-1)/2} \; W_{k+u/2,\, m-u/2}(\rho pt) .$$

Here we see that the function $\phi(\rho pt)$ is finite at zero and at infinity. The function $\phi(\rho pt)$ is continuous within the interval $(0,\infty)$. So the function $\phi(\rho pt)$ is bounded for all positive t.

Hence

$$|\,(pt)^{\sigma-\nu+(u-1)/2} \; W_{k+u/2,\, m-u/2}(\rho pt)\,| \leqslant M_1$$

for all positive t.

Thus (12) becomes

$$|F(p)| \leqslant c M_1 \sup_{x \leqslant t < \infty} \frac{|(1+t^2)^n \sum_{r=0}^{n} \binom{n}{r}}{}$$

$$\lambda^{(n-r)}(t) \sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{r-\nu}$$

$$\cdot \left\{ \frac{1}{2} \ (q - \rho) p - c \right\}^{r-\nu} \sum_{u=0}^{\nu} \binom{\nu}{u} \frac{(\sigma - m) !}{(\sigma - m - \nu + u)!} (-1)^u p^\nu$$

$$\rho^{m + (u - 1)/2} \ . \ e^{-(qp - 2c) (t - x)/2}.$$

$$e^{-(qp - 2c) \ x/2} \mid \cdot \quad \sigma \geqslant m + \nu \geqslant 0 \qquad (13)$$

Now if τ is a fixed number such that $0 < \tau < x$, then

$$\left\{ \frac{1}{2} \ (q - \rho) \ p - c \right\}^{r-\nu} p^\nu \rho^{m + (u - 1)/2} \ e^{-(qp - 2c) \ x/2}$$

$$\leqslant A' \ e^{-(qp - 2c) \tau/2}$$

for $p > 2c/q$. A' being a constant .

Again, for $p > (2c + 1)/q$,

$$e^{-(pq - 2c) \ (t - x)/2} \leqslant e^{-(t - x)/2} \quad \text{for } t \geqslant x \ .$$

So the right-hand side of (13) is dominated by

$$c \ A' \ M_1 \ e^{-(pq - 2c) \ \tau/2} \sup_{x \leqslant t < \infty} \mid (1 + t^2)^n \sum_{r=0}^{n} \binom{n}{r}$$

$$\lambda^{(n-r)} (t) \ . \ \sum_{=0}^{r} \binom{r}{\nu} (- 1)^{r-\nu}$$

$$\sum_{u=0}^{\nu} \binom{\nu}{u} \frac{(\sigma - m) !}{(\sigma - m - \nu + u) !} (- 1)^u \ e^{-(t - x)/2} \mid , \sigma \geqslant m + \nu \ .$$

But the function of t inside the Sup symbol is bounded for all $t \geqslant x$. Consequently , the last expression is less than $M \ e^{-qp\tau/2}$, where M is a sufficiently large positive constant .

Hence $\quad | F(p) | \leqslant M e^{-qp\tau/2}$

for $(1 + 2c)/q < p < \infty$.

Theorem 2 (*cf*. [7, *p*. 354, Theorem 4.2]). *Let $f(t)$ be a generalized Laplace transformable distribution having its support in $0 \leqslant t < \infty$ and let us assume that, over some neighbourhood of the origin, $f(t)$ is a regular distrtbution corresponding to a (Lebesgue) integrable function $h(t)$. Assume that there exist a complex number α and a real number $\eta > -1$ such that*

$$\lim_{t \to 0+} \left\{ \frac{h(t)}{t^{\eta}} \right\} = \alpha \tag{14}$$

Then

$$\lim_{p \to \infty} \left\{ \frac{p^{\eta+1} F(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \right\} = \alpha \tag{15}$$

where $\beta(\eta, m, k, \rho, \sigma, q)$ *and* $F(p)$ *are defined in Theorem* 1 .

Proof . Let the neighbourhood over which $f(t)$ is a regular distribution be $-\infty < t < T$ $(T > 0)$ and let $0 < y < T$. Then $f(t)$ can be decomposed into $f(t) = f_1(t) + f_2(t)$, where the support of $f_1(t)$ is contained in the interval $0 \leqslant t \leqslant y$ and the support of $f_2(t)$ is contained in the interval $y \leqslant t < \infty$.

Now it is clear that $f_1(t)$ is a regular distribution that corresponds to $h(t)$ for $0 < t < y$. Let us suppose that

$$F_1(p) = S^{(\rho, \sigma)}_{q, k, m} [f_1(t) : p] \text{ and } F_2(p) = S^{(\rho, \sigma)}_{q, k, m} [f_2(t) : p]$$

and

$$\frac{P^{\eta+1} F(p)}{\beta(\eta, m, k, \rho, \sigma, q)}$$

$$= \frac{P^{\eta+1} F_1(p)}{\beta(\eta, m, k, \rho, \sigma, q)} + \frac{P^{\eta+1} F_2(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \tag{16}$$

So, by the above lemma,

$$\lim_{P \to \infty} \left\{ \frac{P^{\eta+1} F_2(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \right\} = 0 \tag{17}$$

Again we know that the ordinary generalized Laplace transform is identical to the distributional generalized Laplace transform of the corresponding regular distribution . So from Theorem 1,

$$\lim_{p \to \infty} \left\{ \frac{P^{\eta+1} F_1(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \right\} = \alpha \tag{18}$$

Now from (16), (17), and (18) we have

$$\lim_{p \to \infty} \left\{ \frac{P^{\eta+1} F(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \right\} = \alpha$$

3. A Final Value Theorem

Here we shall relate the behaviour of $f(t)$ as $t \to \infty$ to the behaviour of $F(p)$ as $p \to 0+$ as in

Theorem 3 (*cf* . [7, *p* 355, Theorem 5.1]) . *If a locally integrable function* $f(t)$ *satisfies Condition A, and if there exist a number* α *and a real number* η ($\eta > -1$) *such that*

$$\lim_{t \to \infty} \left\{ \frac{f(t)}{t^{\eta}} \right\} = \alpha , \tag{19}$$

then

$$\lim_{x \to 0+} \left\{ \frac{p^{\eta+1} \; F(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \right\} = \alpha \tag{20}$$

where $\beta(\eta, m, k, \rho, \sigma, q)$ *and* $F(p)$ *are already defined above* .

Proof . From the relation (19) it is clear that $f(t)$ is a function of slow growth and so its generalized Laplace transform has a region of convergence that contains at least the half plane Re $p > 0$. Now let us assume that the support of $f(t)$ is bounded on the left at $t=T$ when $T \geqslant 0$.

Now, using the known result (8) , we have

$$| p^{\eta+1} F(p) - \alpha \beta(\eta, m, k, \rho, \sigma, q) |$$

$$\leqslant p^{\eta+1} \int_0^{\infty} e^{-qpt/2} \quad (pt)^{\sigma - 1/2}$$

$$W_{k,m}(\rho pt) \; | f(t) - \alpha t^{\eta} | \; dt$$

$$\leqslant p^{\eta+1} \int_0^{y} e^{-qpt/2} (pt)^{\sigma - 1/2} \; W_{k,m}(\rho pt) \, | f(t) - \alpha t^{\eta} | \, dt$$

$$+ \; p^{\eta+1} \int_y^{\infty} e^{-qpt/2} \quad (pt)^{\sigma - 1/2} \; W_{k,m}(\rho pt) \, | f(t) - \alpha t^{\eta} | \; dt$$

$$= I_1 + I_2 \text{ , say .} \tag{21}$$

Now let us first consider I_2 ,

$$I_2 = p^{\eta+1} \int_y^{\infty} e^{-qpt/2} (pt)^{\sigma - 1/2} W_{k,m} (\rho pt) t^{\eta} \left| \frac{f(t)}{t^{\eta}} - \alpha \right| dt$$

$$\leqslant \sup_{y \leqslant t < \infty} \left| \frac{f(t)}{t^{\eta}} - \alpha \right| p^{\eta+1}$$

$$\int_0^{\infty} t^{\eta} \; e^{-qpt/2} \; (pt)^{\sigma-1/2} \; W_{k,m} \; (\rho pt) \; dt$$

$$\leqslant \sup_{y \leqslant t < \infty} \left| \frac{f(t)}{t^{\eta}} - \alpha \right| p^{\eta+1}$$

$$\int_0^{\infty} t^{\eta} \; e^{-qpt/2} \; (pt)^{\sigma - 1/2} \; W_{k,m} \; (\rho pt) \; dt$$

$$= \beta (\eta, m, k, \rho, \sigma, q) \sup_{y \leqslant t < \infty} \left| \frac{f(t)}{t^{\eta}} - \alpha \right| \tag{22}$$

Let ε be a fixed arbitrary positive number. From relation (19) it is clear that we ean choose y so large that the right hand side of (22), which is independent of P, can be made less than ε.

Now

$$I_1 = p^{\eta+1} \int_0^{y} e^{-qpt/2} \; (pt)^{\sigma - 1/2} \; W_{k,m} (\rho pt) \, | f(t) - \alpha t^{\eta} | \quad dt$$

Having fixed y, we can choose p so small that integral I_1 becomes less than ε. Hence

$$| p^{\eta+1} F(p) - \alpha \beta (\eta, m, k, \rho, \sigma, q) | \leqslant \varepsilon .$$

This proves (20) for all cases when $T \geqslant 0$.

If $T < 0$, then the additional terms

$$I_3 = \left| \, p^{\eta+1} \int_T^0 e^{-qpt/2} (pt)^{\sigma - 1/2} \, W_{k,m} (\rho pt) \, f(t) \, dt \, \right|$$

occurs on the right-hand side of (21) . Putting

$$G(pt) \equiv e^{-qpt/2} \quad (pt)^{\sigma - 1/2} \; W_{k,m} (\rho pt) \; .$$

We see that the function $G(pt)$ has a fixed value at T and is also finite at zero for $m > 0$. The function $G(pt)$ is continuous within this interval . Hence the function $G(pt)$ must be bounded in the interval $(T, 0)$, $T < 0$. So let $G(pt)$ attain its maximum in the interval $(T, 0)$ at the point $t = \zeta$.

Thus

$$I_3 = \left| \, p^{\eta+1} \, e^{-qp\,\zeta/2} \quad (p\zeta)^{-1/2} \; W_{k,m} \, (\rho p \zeta) \, \right| \; \int_T^0 | \, f(t) \, | \, dt$$

Now I_3 approaches zero as $p \to 0+$, so that

$$| \, p^{\eta+1} \; F(p) - a \, \beta \, (\, \eta, m, k, \rho, \sigma, q \,) \, | \leqslant \varepsilon$$

for all values of T . Theorem 3 is thus proved .

Now we extend Theorem 3 to certain distributions , and we have **Theorem 4** (*cf* . [7, p. 357, Theorem 5.2]) . *Let $f(t)$ be a generalized Laplace transformable distribution in D'_R and let us assume that , over some semi-infinite interval $\tau < t < \infty$. $f(t)$ is a regular distribution corresponding to a locally integrable function $h(t)$ that satisfies the second part of Condition A . Also let us assume that there exist a complex number a and a real number η ($\eta > -1$) such that*

$$\lim_{t \to \infty} \left\{ \frac{h(t)}{t^{\eta}} \right\} = a \qquad . \tag{23}$$

Then

$$\lim_{p \to 0+} \left\{ \frac{p^{\eta+1} \; F(p)}{\beta(\eta, m, k, \rho, \sigma, q)} \right\} = \alpha \tag{24}$$

where $F(p)$ *and* $\beta(\eta, m, k, \rho, \sigma, q)$ *are defined as before* (*p is real*) .

Proof . Here $f(t)$ can be decomposed into

$$f(t) = f_1(t) + f_2(t) ,$$

where $f_2(t)$ has its support contained in $y \leqslant t < \infty$ $(y > \tau)$ and $f_1(t)$ has a bounded support that does not extend to the right of $t = y$.

Thus

$$F_1(p) = S_{q,\,k,\,m}^{(\rho,\,\sigma)} \left[f_1(t) : p \right]$$

in an entire function of p, so that

$$\lim_{p \to 0+} \left\{ p^{\eta+1} \; F_1(p) \right\} = 0 ,$$

Therefore, we have

$$\lim_{p \to 0+} \left\{ p^{\eta+1} \; F(p) \right\} = \lim_{p \to 0+} \left\{ p^{\eta+1} \; F_2(p) \right\} \tag{25}$$

Now from Theorem 3 the right-hand side of (25) equals to $\alpha\beta(\eta, m, k, \rho, \sigma, q)$. That is,

$$\lim_{p \to 0+} \left\{ p^{\eta+1} \; F(p) \right\} = \alpha\beta(\eta, m, k, \rho, \sigma, q) ,$$

so that

$$\lim_{p \to 0+} \left\{ \frac{p^{\eta + 1} F(p)}{\beta(\eta, m, k, \rho. \sigma, q)} \right\} = \alpha$$

which proves the theorem .

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ON (-1,1) RINGS WITH COMMUTATORS IN THE LEFT NUCLEUS

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ABSTRACT

In this paper we show that a left primitive $(-1,1)$ ring with commutators in left nucleus is either associative or simple with right identitiy element .

1. INTRODUCTION

A $(-1,1)$ ring R is a non-associative ring in which the following identities hold :

(1) $(x,y,z)+(x,z,y)=0$

(2) $(x,y,z)+(y,z,x)+(z,x,y)=0$

for all x,y,z belonging to R where $(x,y,z)=(xy)z-x(yz)$.

The left nucleus LN is defined as the set :

$LN=\{x \in R : (x,y,z)=0 \text{ for all } y,z \in R\}$.

Throughout this paper, we assume R to be a $(-1,1)$ ring of characteristic $\neq 2, \neq 3$ such that $((x,y),z,w)=0$ for all $x,y,z,w \in R$ where the commutator $(x,y)=xy-yx$.

That is, $(R,R) \subseteq LN$.

Let $U = \{ u \in R : (u,x) = 0$ for all $x \in R \}$. Then by virtue of $((R, R) , x,y) = 0$ for all $x, y \in R$ we have

$$(3) \quad (R,R,R) \subseteq U$$

where (R,R,R) is the set of all finite sums of elements of the form (x,y,z) . This is equation (11) in [4] .

The following identity given in [1] holds in a $(-1,1)$ ring of characteristic $\neq 2 , \neq 3$.

$$(4) \quad (w,xy,z) + (w,zy,x) - (w,y,z)\, x - (w,y,x)\, z = 0$$

for all $w,x,y,z \in R$.

2. CONSTRUCTION OF IDEALS

Lemma 1. *For any ring R, $(R,R,R) + R\,(R,R,R)$ is a two - sided ideal of R .*

This is proved in [3] .

Lemma 2. *Let A be a left ideal of R. Then*

(a) $S = \{ s \in A : sR \subseteq A \}$ *is a two - sided ideal of* R,

(b) $(R,A,R) \subseteq S$.

Proof. (a) For any $s \in S$ and $x,y \in R$

$$\begin{aligned}(sx)\, y &= (s,x,y) + s\,(xy)\\ &= -(x,y,s) - (y,s,x) + s\,(xy) \quad \text{(using (2))}\\ &= -(x,y,s) + (y,x,s) + s\,(xy) \quad \text{(using (1))}\end{aligned}$$

A is a left ideal and $s \in S$ implies

$-(x,y,s) + (y,x,s) + s(xy) \in A$. Hence $(sx)y \in A$. Also $sR \subseteq A$ implies $sx \in A$. Hence $sx \in S$. Now,

$$(xs)y = (x,s,y) + x(sy)$$

$$= -(x.y,s) + x(sy) \quad (\text{using } (1))$$

Again, using the fact that A is a left ideal and $s \in S$ in the above equation, we get $(xs)y \in A$. Hence $xs \in S$. So S is a two-sided ideal of R.

(b) Let $x,w,z \in R$ and $a \in A$. *Then*

$$(w,a,x)z = -(w,x,a)z \qquad (\text{using } (1))$$

$$= -z(w,x,a) \qquad (\text{using } (3))$$

Again, since A is a left ideal, $-z(w,x,a) \in A$. Therefore, $(w,a,x)z \in A$, Hence $(R,R,R) \subseteq S$.

Theorem 1. *If R has a maximal left ideal $A \neq (0)$ which contains no two-sided ideal of R other than (0), then R is associative.*

Proof. By Lemma 2, S is a two-sided ideal of R contained in A. Hence $S = (0)$. Since $(R,A,R) \subseteq S$, we have $(R,A,R) = (0)$. On the other hand it is easy to verify that $A+AR$ is a two-sided ideal of R. Since $A \subset A+AR$ we must have $A+AR = R$. Hence

$$(R,R,R) = (A+AR,R,R)$$

$$\subseteq (A,R,R) + (AR,R.R)$$

$$\subseteq (A,R,R) + (RA,R,R) \qquad (\text{since } ((R,R),R,R) = (0))$$

$$\subseteq (A,R,R) + (A,R,R) \qquad (\text{since } RA \subseteq A)$$

$$\subseteq (A,R,R)$$

But $(R,A,R) = (0)$ implies $(R,R,A) = (0)$ (using (1)),

and so $(A,R,R) = (0)$ (using (2)).

Therefore, $(R,R,R) \subseteq (0)$. Hence R is associative.

3. LEFT PRIMITIVE RINGS

Definition 1. A left ideal A of R is called regular if there exists an element $g \in R$ such that $x - xg \in A$ for all $x \in R$.

Definition 2. A ring R is called left primitive if it (contains a regular maximal left ideal which contains no two-sided ideal of R other than (0).

Theorem 2. *If R is a left primitive ring, then either R is associative or it is simple with a right identity element.*

Proof. Let A be a regular maximal left ideal which contains no two-sided ideal of R other than (0). If $A \neq (0)$, then by Theorem 1, R is associative. Thus assume that (0) is a maximal regular left ideal of R, which contains no two-sided ideal of R other than (0). (0) is maximal implies, R has no proper ideal. Since (0) is regular, there exists $g \in R$ such that $x - xg \in (0)$ for all $x \in R$. That is, $x = xg$ for all $x \in R$. This implies $RR \neq (0)$ and g is a right identity element. Hence R is simple with a right identity element.

Finally, at the referee's suggestion, we add a theorem which was proved in [4]. This theorem states: *If these $(-1,1)$ rings are semiprime, then they are associative*. This result follows easily, since in these rings $(R,R,R)^2 = (0)$: For the sake of completeness, the proof of this theorem may be recalled here as outlined below (*cf*. [4]):

(a) (R,R,R) is an ideal ;

(b) If $u \in U$, then $(x,x,y)\,u = (x,x,yu)$;

(c) If $u \in U$, then $(x,x,u) = 0$;

(d) Since $(R,R,R) \subseteq U$ and (R,R,R) is an ideal, then

$$(x,x,R)\ (R,R,R) \subseteq (x,x,R\ (R,R,R) \subseteq (x,x,U) = 0\ ;$$

(e) $(R,R,R) \subseteq \{ (x,x,R) : x \in R \}$.

Thus, by (d) and (e), $(R,R,R)\ (R,R,R) = 0$.

These identities are all given in [2] .

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CERTAIN MULTIDIMENSIONAL GENERATING RELATIONS

By

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ABSTRACT

In [1] we defined a general polynomial system $S_n^A [(x_m), y]$ with the help of a generating relation involving the Lauricella function F_A . In the present paper we derive a linear and a bilateral generating relation for these polynomials . Finally, we apply our results to derive several multilinear and multilateral generating relations .

1. INTRODUCTION

The general polynomial system $\left\{ S_n^A [(x_m) , y] / n = 0, 1,2 \right\}$

is defined by means of the generating relation [ ] :

$$\sum_{n=0}^{\infty} S_{n;\nu, (r_m;1) , (c_m)}^{A;m,a, (b_m), (\mu_m)} [(x_m) , y] t^n$$

$$e^{\nu yt} F_A [a, (b_m) ; (c_m) ; \mu_1 x_1 t, \mu_2 x_2 t^{r_2} ,..., \mu_m x_m t^{r_m}], \tag{1.1}$$

where F_A is one of the Lauricella functions [4,p.41] .

The conditions for validity of (1.1) are :

(i) (x_m), y, t are all finite quantities ;

(ii) m, $(r_{m;1})$ are natural numbers ;

(iii) ν, (μ_m) are finite numbers ;

(iv) a, (b_m) are non-zero numbers ;

(v) (c_m) are neither zero nor negative integerals .

For this polynomial system, we have an explicit representation in the form [1] :

$$\overset{A}{\underset{n}{S}} [(x_m), y] = \sum_{m_1=0}^{n} \sum_{m_2=0}^{[(n-m_1)/r_2]} \cdots \sum_{m_m=0}^{[(n-m_1-\sum_2^m r_i m_i)/r_m]} \frac{(a)_{n-m_1-\sum_2^m (r_i-1)m_i} \, (b_1)_{n-m_i-\sum_2^m r_i m_i} \, [(b_{m;1})_{(m;1)}]}{(n-m_1-\sum_2^m r_i m_i)! \, (c_1)_{n-m_1-\sum_2^m r_i m_i} \, [(c_{m;1})_{(m;1)}]}$$

$$\frac{(xy)^{m_1} \, (\mu_1 x_1)^{n-m_1-\sum_2^m r_i m_i} \, [(\mu_{m;1} x_{m;1})^{m_{m;1}}]}{[(m_m!)]} \qquad (1.2)$$

which we shall use here .

Generating relations play an important role in the study of polynomial sets . Recently, Srivastava and Manochn [9] (see also McBride [5] have discussed a numper of useful methode of obtaining generating functions . L. Carlitz and H. M. Srivastava ([2] and [3]), R. Panda ([6] and [7]), A, B. Rao and Anura Srivastava [8], and

some others (see [9] have studied polynomials of the form $F_1, F_2, F_D^{(r)}$ and F_4 with the help of generating relations .

In this paper we derive a linear and a bilateral generating relations, which ultimately yields as its applications , various multidimensional generating relations .

We employ the following notations in our present investigations :

(i) $(m) = 1,2,\ldots, m$.

(ii) $(a_p) = a_1, a_2, \ldots, a_p$.

(iii) $[(a_p)]_n = (a_1)_n (a_2)_n \ldots (a_p)_n$.

(iv) $[(a_{m;i})_{(p_{m;i})}] = (a_1)_{p_1} (a_2)_{p_2} \ldots (a_{i-1})_{p_{i-1}}$

$$(a_{i+1})_{p_{i+1}} \ldots (a_m)_{p_m}$$

(v) $\triangle (r;a) = \dfrac{a}{r}, \dfrac{a+1}{r}, \ldots, \dfrac{a+r-1}{r}$.

(vi) $[(a_m x_m)^{p_m}] = (a_1 x_1)^{p_1} = (a_2 x_2)^{p_2} \ldots (a_m x_m)^{p_m}$.

(vii) $[(a_{m;i}\ x_{m;i})^{p_{m;i}}] = (a_1 x_1) \ldots (a_{i-1} x_{i-1})^{p_{i-1}} (a_{i+1} x_{i+1})^{p_{i+1}} \ldots$
$(a_m x_m)^{p_m}$.

2. LINEAR GENERATNG RELATION

Using the series form, we finally have

$$\sum_{n=0}^{\infty} \frac{[(\alpha'_N)]_n [(\alpha''_{N'})]_n}{[(\beta_M)]_n} S_n^A [(x_m), y] t^n$$

$$= F \begin{matrix} N+N'+1 : 0; 1; \dots ; 1 \\ M : 0; 1; \dots ; 1 \end{matrix} \left[\begin{matrix} [\,(\alpha'_N) : 1,\ r_2 \dots, r_m\,], \\ [\,(\beta_M) ; 1,\ r_2 \dots, r_m\,], \end{matrix} \right.$$

$$[\,(\alpha''_{N'}) : 1, r_2, \dots, r_m\,],\ (a : 1, 0, 1, \dots, 1) : -; (b_1 : 1); \dots; (b_m : 1);$$

$$\underline{\qquad\qquad\qquad\qquad}\ , : -; (c_1 : 1); \dots; (c_m : 1);$$

$$\left. \nu y t,\ \mu_1 x_1 t,\ \mu_2 x_2 t^{r_2}, \dots, \mu_m x_m t^{r_m} \right]. \tag{2.1}$$

It may also be written as

$$\sum_{n=0}^{\infty} \frac{[\,(\alpha'_N)\,]_n\, [\,(\alpha''_{N'})\,]_n}{[\,(\beta_M)\,]_n}\ S_n^A\,[\,(x_m),\ y\,]\, t^n$$

$$\sum_{n=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_{m-1}=0}^{\infty} \frac{[\,(\alpha'_N)\,]_{n+m_1+\sum_2^{m-1} r_i m_i}}{[\,(\beta_M)\,]_{n+m_1+\sum_2^{m-1} r_i m_i}}$$

$$\frac{[\,(\alpha''_{N'})\,]_{n+m_1+\sum_2^{m-1} r_i m_i}}{\qquad} \ \frac{(b_1)_n\ (a)_{n+m_2+\dots+m_{m-1}}}{(c_1)_n\, n!}$$

$$\frac{[\,(b_{m-1;1})_{(m_{m-1;1})}\,]}{[\,(m_m!)\,]\ [\,(c_{m-1;1})_{(m_{m-1;1})}\,]} \frac{(\nu y)^{m_1}\, (\mu_1 x_1)^n}{\qquad} \frac{[\,(\mu_{m-1;1}\ x_{m-1;1})}{\qquad}$$

$$t^{n+m_1+\sum_{2}^{m-1} r_i m_i} \; {}_{r_m (N+N')+2}F_{r_m M+1}$$

$$\left[\begin{array}{l} \Delta(r_m; (\alpha'_N) + n + m_1 + \sum_{2}^{m-1} r_i m_i), \\ \\ \Delta(r_m; (\beta_M) + n + m_1 + \sum_{2}^{m-1} r_i m_i), \end{array}\right.$$

$$\left[\begin{array}{l} \Delta(r_m; (\alpha''_{N'}) + n + m_1 + \sum_{2}^{m-1} r_i m_i), \; a+n+m_2+ \ldots +m_{m-1}, \; b_m; \\ \\ \rule{4cm}{0.4pt}\;, \quad \rule{3cm}{0.4pt}\;, \quad c_m\,; \end{array}\right.$$

$$\left. g_m x_m t^{r_m} \right],$$

$$\text{where } g_m = \mu_m r_m^{(N+N'-M) r_m} . \tag{2.2}$$

3. BILATERAL GENERATING RELATION

With the help of the generalized Lauricella function of Srivastava and Daoust (*cf.* [4] (and using (2.2) , we finally obtain the following bilateral generating relation for our polynomial system :

$$\sum_{n=0}^{\infty} \frac{[(\alpha'_M)]_n \, [(\alpha''_M)]_n}{[(\beta_N)]_n} S_n^A [(x_m), y]$$

$$F^{A' : B^1 + 2 ; \ldots ; B^{(M)} + 2}_{c : D^1 ; \ldots ; D^{(M)}}$$

$$\left[\begin{array}{l} [(\delta) : 1, \ldots, 1] : (a_1' + n : 1), (\alpha_1'' + n : 1), [(b') : 1] ; \ldots \\ [(\gamma) : 1, \ldots, 1] : \qquad\qquad\qquad\qquad [(d') : 1] ; \ldots \end{array}\right.$$

$$; (\alpha'_M + n : 1), (\alpha_M'' + n ; 1), \left.\begin{matrix} [(b^M) : 1] ; \\ [d^M) : 1] ; \end{matrix} z_1, \ldots, z_M \right] t^n$$

$$= \sum_{n=0}^{\infty} \sum_{m_1=0}^{\infty} \cdots \sum_{m_m=0}^{\infty}$$

$$\frac{[(\alpha'_M)]_{n+m_1+\sum_2^m r_i m_i} \, [(\alpha''_M)]_{n+m_1+\sum_2^m r_i m_i} \, (a)_{n+m_2+\ldots+m_m}}{[(\beta_N)]_{n+m_1+\sum_2^m r_i m_i} \; n\,!\,(c_1)_n\,[(m_m\,!)]}$$

$$\frac{(b_1)_n\,[(b_{m;1})_{(m_{m;1})}]}{[(c_{m;1})_{(m_{m;1})}]} (\mu_1 x_1 t)^n \, (\nu y t)^{m_1} \, [(\mu_{m;1} x_{m;1} \, t^{r_{m;1}})^{m_{m;1}}] .$$

$$F^{A' : B^1 + 2 ; \ldots ; B^{(M)} + 2}_{C : D^1 ; \ldots ; D^{(m)}}$$

$$\left[\begin{matrix} [(\delta) : 1, \ldots, 1] : (\alpha_1' + n + m_1 + \sum_2^m r_i m_i : 1), (\alpha_1'' + n + m_1 + \sum_2^m r_i m_i : 1), \\ \hline [(\gamma) : 1, \ldots, 1] : \end{matrix}\right.$$

$$[(b') : 1] ; \ldots ; (\alpha'_M + n + m_1 + \sum_2^m r_i m_i : 1), \alpha''_M + n + m_1 + \sum_2^m r_i m_i : 1), [(b^{(M)}) : 1] ;$$

$$\left.[(d') : 1] ; \ldots ; \qquad \text{———————} \qquad , [(d^M) : 1] ; \quad z_1, \ldots, z_M \right] \qquad (3\ 1)$$

4. APPLICATIONS

Formula (3.1) with $\nu = 0$, $m = 2$, $= r_2 = \mu_1$, $(b_2) = (c_2)$, $a = 1/2$,

$x_1 = x,\ \mu_2 = -1,\ x_2 = 1,\ \alpha_j' = 1,\ \alpha_j'' = \beta_j = 1,\ z_j = \frac{1-z_j}{1+z_j}$

$(j=1,\dots M)$, $B_1 = \dots = B^{(M)} = 0,\ D^1 = \dots = D^{(M)} = 1,\ A' = C,\ \delta=\gamma$

yields :

$$\sum_{n=0}^{\infty} (n!)^M P_n(x) P_n(z_1) \dots P_n(z_M) \left\{ \frac{2}{(1+z_1)} \cdots \frac{2}{(1+z_M)} \right\}^{-nM+1} t^n$$

$$= \sum_{n,m_1=0}^{\infty} \frac{\left\{ (1)_{n+2m_1} \right\}^M (1/2)_{n+m_1} (2xt)^n (-t^2)^{m_1}}{n!\ m_1!}$$

$$\prod_{j=1}^{M} {}_2F_1\left[\begin{matrix} 1+n+2m_1,\ 1+n+2m_1\ ; \\ 1\ ; \end{matrix} -\frac{1-z_j}{1+z_j}\right] , \tag{4.1}$$

which is a multilinear generating relation for Legendre polynomials .

By specializing various parameters in (3.1) , we have :

$$\sum_{n=0}^{\infty} \frac{\left\{ (1+2\alpha)_n \right\}^{M+1} \left\{ (1+\alpha)_n \right\}^{M-1}}{(n!)^M} \left\{ \frac{2}{(1+z_1)} \cdots \frac{2}{(1+z_m)} \right\}^{-(1+2\alpha+n)}$$

$$\cdot\ P_n^{(\alpha,\alpha)}(x) P_n^{(\alpha,\alpha)}(z_1) \dots P_n^{(\alpha,\alpha)}(z_M)\ t^n$$

$$= \sum_{m_1,n=0}^{\infty} \frac{\left\{ (1+2\alpha)_{n+m_1} \right\}^M \left\{ (1+\alpha)_n \right\}^{M-1}}{n!\ m_1!\ (1)_{n+2m_1}} (2xt)^n (-t^2)^{m_1}$$

$$\prod_{j=2}^{M} {}_2F_1\left[\begin{matrix}1+2\alpha+n+2m_1, & 1+\alpha+n+2m_1; & -\dfrac{1-z_j}{1+z_j}\\ & 1+\alpha \quad ; & \end{matrix}\right]. \tag{4.2}$$

With suitable conditions, we have :

$$\sum_{n=0}^{\infty} \frac{\left\{(2)_n\right\}^M \left\{(3/2)_n\right\}^M}{\left\{(1)_n\right\}^M} \left\{\left(\frac{2}{1+z_1}\right)\cdots\left(\frac{2}{1+z_M}\right)\right\}^{-(2+n)}$$

$$\cdot\, U_n(x)\, U_n(z_1) \ldots U_n(z_M)\, t^n$$

$$= \sum_{n,m_1=0}^{\infty} \frac{\left\{(2)_{n+2m_1}\right\}^M \left\{(3/2)_{n+2m_1}\right\}^M}{n!\, m_1!\left\{(1)_{n+2m_1}\right\}^M} (1)_{n+m_1}$$

$$\cdot \prod_{j=1}^{M} \left[\begin{matrix}2+n+2m_1, \; 3/2+n+2m_1 \;; & -\dfrac{1-z_j}{1+z_j}\\ 3/2 \quad ; & \end{matrix}\right] (2xt)^n(-t^2)^{m_1} \tag{4.3}$$

which is a multilinear generating relation for Tchebycheff polynomials.

Various other applications have similarly been deduced. Several known bilinear, bilateral, trilinear, trilateral generating relations happen to be special cases of our results (4.1), (4.2) and (4.3).

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DIFFERENTIAL EQUATIONS FOR A GENERAL CLASS OF POLYNOMIALS

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ABSTRACT

The object of the present paper is to derive the differential equations satisfied by a general class of polynomials generated by (1 1) below. We also present a brief discussion about the complete solution and verification of the solution for one of the various cases involved.

1. INTRODUCTION

In the present paper we derive the differential equations for a general class of polynomials which provide a generalization of a number of orthogonal and other polynomial systems. This general polynomial set is defined by a generating relation (see [2], [3] and [4]) :

$$(1-v_1\, x^{c_1} y^{c_2}\, t^{a_1})^{-\lambda}\; F^{\,r\,:\,p\,;\,u}_{\,s\,:\,q\,;\,v}\left[\begin{matrix}(a_r) : (a_p^*) ; (\alpha_u^*) ; \\ (b_s) : (b_q^*) ; (\beta_v^*) ;\end{matrix}\right.$$

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$$\frac{\mu\, x^{c_1'}\, t^{a_2}}{(v_1 x^{c_1}\, y^{c_2}\, t^{\alpha_1}-1)^{m_1}}, \quad \frac{v y^{c_2}\, t^{\alpha_3}}{(1-v_1\, x^{c_1}\, y^{c_2}\, t^{a_1})^{m_2}} \Bigg]$$

$$= \sum_{n=0}^{\infty} A^{\lambda ; \mu ; v ; c_1' ; c_2' ; \alpha_2 ; \alpha_3 ; (a_r) ; (a_p^*) ; \alpha_u^*)}_{\alpha_2 n ; v_1 ; c_1 ; c_2 ; m_1 ; m_2 ; a_1 ; b_s) ; b_q^*) ; (\beta_v^*)} (x,y) t^{\alpha_2 n} \quad (1\cdot 1)$$

where $F^{r:p;u}_{s:q;v}$ is a general double hypergeometric function (*cf.*, *e.g*, Srivastava and Manocha [6, *p*. 63 ; *Eq*. (16) *et seq*.] ; see also Burchnall and Chaundy [1, p. 112]).

The above general polynomial set contains a number of parameters; therefore, for the sake of simplicity, we shall denote it by $A_{\alpha_2 n}(x, y)$, unless there is some change in any one of the parameters (except $\alpha_2 n$) which is the order (degree) of the polynomial set. If there be any change in any of the parameters, only the changed parameter will be indicated. For example

$$A^{\lambda ; \mu+1 ; v ; c_1' \ldots}_{a_2 n ; v_1 ; 0 \ldots}(x,y) = A^{\mu+1}_{a_2 n ; v_1 ; 0}(x,y)$$

The explicit form of the polynomial set generated by (1 1) is

$A_{\alpha_2 n}(x,y)$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{[(a_r)]_{n-r_1 k-d_2 l}\, [(a_p^*)]_{n-d_1 k-d_2 l}\, [(\alpha_u^*)]_k}{[(b_s)]_{n-r_1 k-d_2 l}\, [(b_q^*)]_{n-d_1 k-d_2 l}\, [(\beta_v^*)]_k} \times$$

$$\frac{\mu^{n-d_1 k-d_2 l}\ x^{c_1'(n-d_1k-d_2l)+c_1 l}\ y^{-c_2'k+c_2 l}\ \nu^{k}\nu_1^{l}\ (\lambda)_{m_1n+D_1k-D_2l}}{(-1)^{-m_1(n-d_1k-d_2l)}\ k!\ l!\ (n-d_1k-d_2l)\ !\ (\lambda)_{m_1n+D_1k-m_1d_2l}} \tag{1.2}$$

where

$d_1 = a_3/a_2,\ d_2 = \alpha_1/a_2,\ r_1 = (d_1-1)$

$D_1 = m_2-m_1d_1$ and $D_2 = m_1d_2-1$,

d_1 and d_2 being non-negative integers (see also [2],[3] and[4] for various notations and conventions used here) .

2. THE GENERAL DIFFERENTIAL EQUATIONS

The differential equations satisfied by the polynomials $A_{a_2n}\ (x,y)$ are contained in the following theorems :

THEOREM 1, *For $D_1 > 0$, and $D_2 = 0 = \nu$, we have the following differential equation with respect to* x :

$$\Big\{\frac{1}{e_1}(\theta-c_1'n)\ [\ (a_r)+n+d_2-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]\ [\ (a_p^*)+n+d_2-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]-\frac{\nu_1x^{e_1}\ y^{c_2}}{\mu^{d_2}}[\ (b_s)+n-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]$$

$$[\ (b_q^*)+n-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]\ [\ 1+n-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]$$

$$[\ 1-\lambda-m_1n+\frac{1}{e_1}(\theta-c_1'n)\ ]\Big\}\ A^*_{a_2n}\ (x,y)=0\ , \tag{2.1}$$

where $e_1 = c_1-c_1'd_2$, $\theta = x\frac{d}{dx}$, *and (for convenience)*

$$A\begin{matrix}\lambda\,;\,\mu\,;\,v\,;\,c_1'\,;\,\ldots\,;\,a_2\,;\,\ldots\,;\,(a_r)\,,\,(a_p{}^*)\,;\,\ldots\\ a_2 n\,;\,v_1\,;\,c_1\,;\,c_2\,;\,m_1\,;\,\ldots\,;\,a_1\,;\,(b_s)\,;\,(b_q{}^*)\,;\,\ldots\end{matrix}(x,y) = A^*_{a_2 n}\ (x,y) \tag{2.2}$$

Theorem 2. *For* $D_1 > 0$ *and* $D_2 = 0 = v$ *, we have the following differential equation with respect to* y :

$$\{ \Psi [(a_r) + n + d_2 - d_2\Psi - (d_2)] [(a_p{}^*) + n + d_2 - d_2\Psi - (d_2)]$$

$$- \frac{v_1 x^{e_1}\ y^{c_2}}{\mu^{d_2}} [(b_s) + n - d_2\Psi - (d_2)] [(b_q{}^*) + n - d_2\Psi - (d_2)]$$

$$[1 + n - d_2\Psi - (d_2)] [1 - \lambda - m_1 n + \Psi] \} A^*_{a_2 n}\ (x,y) = 0\ , \tag{2.3}$$

where $\Psi = \dfrac{y}{c_2}\ \dfrac{d}{dy}$

Proof . The proofs of Theorem 1 and Theorem 2 are fairly straightforward, and we omit the details .

The complete solution of (2.1) and (2.3) are given by (see Rainville [5])

THEOREM 3. *For* $D_1 > 0$ *and* $D_2 = 0 = v$ *, the complete solutions of the differential equations* (2.1) *and* (2.3) *are*

$$W = N_0 L_0 + \sum_{g=1}^{d_2} \sum_{h=1}^{r} N^{(1)}_{g,h} L^{(1)}_{g\,h} + \sum_{g=1}^{d_2} \sum_{u=1}^{p} N^{(1)}_{g,u} L^{(1)}_{g,u} \tag{2.4}$$

and

$$W^* = N_0{}^* L_0 + \sum_{g=1}^{d_2} \sum_{h=1}^{r} N^{(2)}_{g,h} L^{(2)}_{g,h} + \sum_{g=1}^{d_2} \sum_{u=1}^{p} N^{(2)}_{g,u} L^{(2)}_{g,u}\ . \tag{2.5}$$

respectively, where N_0, $\overset{(1)}{N}_{g,h}$, $\overset{(1)}{N}_{g,u}$ *and* N_0^*, $\overset{(2)}{N}_{g,h}$, $\overset{(2)}{N}_{g,u}$

are arbitrary functions of y and x, respectively, and

$$L_0 = A^*_{a_2 n}\ (x,y)\ ,\ \overset{(1)}{L}_{g,h} = x^k\ {}_mF_n\ [R_1]$$

$$\overset{(1)}{L}_{g,u} = x^{k_1}\ {}_mF_n\ [R_2]\ ,\ \overset{(2)}{L}_{g,h} = y^{k_2}\ {}_mF_n\ [R_1]$$

$$\overset{(2)}{L}_{g,u} = y^{k_3}\ {}_mF_n\ [R_2]\ ,\ \text{where } k = \frac{e_1}{d_2}((a_h) + n + d_2 - g) + c_1'n\ ,$$

$$k_1 = \frac{e_1}{d_2}((a_u^*) + n + d_2 - g) + c_1'n\ ,\ k_2 = \frac{1}{d_2}((a_h) + n + d_2 - g),$$

$k_3 = \dfrac{1}{d_2}((a_v^*) + n + d_2 - g)$, *and* R_1 *and* R_2 *are defined as follows :*

$$R_1 = F\left[\begin{array}{l} 1,\ \triangle(d_2\ ;\ 1-g-(b_s)+a_h+d_2)\ ,\ \triangle(d_2\ ;\ a_h+d_2-g)\ , \\ \triangle(d_2\ ;\ 1-g-(b_1^*)+a_h+d_2)\ ,\ (2-\lambda-m_1 n+(o_h+n-g)/d_2)\ ; \\ \qquad\qquad\qquad\qquad \dfrac{\mu^{-d_2}\ v_1\ x^{e_1}\ y^{c_2}}{(-d_2)^{d_2(p-q+r-s-1)}} \\ \triangle(d_2\ ;\ 1-g-(a_r)+a_h+d_2),\triangle(d_2;\ 1-g-(a_p^*)+a_h+d_2)\ , \\ ((a_h+n-g)/d_2+2)\ ; \end{array}\right]$$

and

$$R_2 = F\left[\begin{array}{l} 1,\ \triangle\,(d_2\ ;\ 1-g-(b_s)+a_u^* + d_2)\ ,\ \triangle\,(d_2\ ;\ a_u^* + d_2 - g)\ , \\ \triangle\,(d_2;\ 1-g-(b_p^*)+a_u^*+d_2),\ (2-\lambda-m_1 n+(a_u^*+n-g)/d_2); \\ \qquad\qquad\qquad\qquad \dfrac{\mu^{-d_2}\ \nu_1 x^{e_1}\ y^{c_2}}{(-d_2)^{d_2\,(p-q+r-s-1)}} \\ \triangle(d_2;\ 1-g-(a_p^*)\ +a_u^*+d_2), \triangle(d_2;\ 1-g-(a_r)+a_u^*+d_2), \\ (\,(a_u^*+n-g)/d_2+2)\ ; \end{array}\right]$$

Finally, we give

THEOREM 4. *For $D_1 > 0$, $D_2 > 0$ and $v > 0$, we have the differential equations with respect to x and y :*

$$\{\frac{1}{e_1}\ (\theta-c_1'n)\ [\ (a_r)+n+d_2-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]\ [\ (a_p^*)+n+d_2-$$

$$\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]\times$$

$$[\lambda+m_1 n+D_2-\frac{D_2}{e_1}(\theta-c_1'n)-(D_2)\ ]-\frac{\nu_1 y^{c_2}\ x^{e_1}}{(-1)^{m_1 d_2}\ \mu^{d_2}}\ [\ (a_r)+n-$$

$$\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]\times$$

$$[\ (b_q^*)+n-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]\ [\ 1+n-\frac{d_2}{e_1}(\theta-c_1'n)-(d_2)\ ]$$

$$[\lambda+m_1 n-(m_1 d_2/e_1)(\theta-c_1'n)-(m_1 d_2)\ ]\ \}\quad A^*_{a_2 n}\ (x,y)=0 \tag{2.6}$$

and

$$\{\Psi\,[\,(a_r) + n + d_2 - d_2\Psi - (d_2)\,]\ [\,(a_p^*) + n + d_2 - d_2\,\Psi - d_2)\,]$$

$$[\,\lambda + m_1 n + D_2 - D_2\,\Psi - (\,D_2\,)\,]$$

$$-\ \frac{\nu_1 y^{c_2}\ x^{e_1}}{(-1)^{m_1 d_2}\ \mu^{d_2}}\,[\,(b_s) + n - d_2\,\Psi - (\,d_2\,)\,]\,[\,(\,b_q^*\,) + n - d_2\,\Psi - (d_2)\,]$$

$$[\,1 + n - d_2\Psi - (d_2)\,]\ \ [\,\lambda + m_1 n - m_1 d_2\Psi - (m_1 d_2)\,]\,\}\ A^*_{\alpha_2 n}\ (x.y) = 0 \quad (2.7)$$

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SLOW UNSTEADY FLOW IN A POROUS ANNULUS WITH AXIAL WAVINESS

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ABSTRACT

Slow unsteady flow of a viscous incompressible fluid between two porous coaxial circular cylinders with axial waviness, under the influence of pulsating pressure gradient has been investigated. The waviness has been taken to be small in comparision to the smooth radii of the circular cylinders. The Fourier integral transform technique has been used for finding the expressions for axial velocity, radial velocity and pressure in terms of modified Bessel functions. For a particular case of sinusoidal waviness, the velocities have been calculated and shown graphically ,

1. INTRODUCTION

Viscous fluid flow through porous wavy boundaries is of practical importance , In aircraft propulsion systems, porous tubes are used as injectors to increase the engine efficiency and as silencers in exhaust systems . Several authors, (namely, Citron [2], Khamrui [7], Verma and Gaur [10] and Gaur and Mehta [4]) have studied the slow viscous fluid flow through rough circular tubes and rough coaxial infinite cylinders . Bauer [1] and Varshney [9] have investigated the puls atile flow

in a circular pipe with reabsorption across the wall and fluctuating flow through a porous medium bounded by a porous plate, respectively. Hepworth and Rice ([5], [6]) have studied the flow between parallel plates and circular / rectangular tubes with arbitrary time varying pressure gradient. Recently, Gaur and Bhatnagar [3] have discussed the slow unsteady flow in a porous tube with axial waviness.

The present investigation deals with slow unsteady flow of a viscous incompressible fluid between two porous coaxial circular cylinders with axial waviness, under the influence of pulsating pressure gradient. The waviness is taken to be small in comparison to the smooth radii of the cylinders. Expressions for the velocity components and pressure are obtained in terms af modified Bessel functions using Fourier transform technique. A particular case of sinusoidal roughness is numerically investigated.

2. PROBLEM FORMULATION

In cylindrical polar coordinates, the Navier - Stokes equations in the non - dimensional form, under the slow flow assumption, are :

$$\frac{\partial U}{\partial T} = -\frac{\partial P}{\partial \lambda} + \frac{\partial^2 U}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial U}{\partial \lambda} + \frac{\partial^2 U}{\partial Z^2} - \frac{U}{\lambda^2} \qquad (1)$$

$$\frac{\partial W}{\partial T} = -\frac{\partial P}{\partial Z} + \frac{\partial^2 W}{\partial \lambda^2} + \frac{1}{\lambda}\frac{\partial W}{\partial \lambda} + \frac{\partial^2 W}{\partial Z^2}, \qquad (2)$$

and

$$\frac{1}{\lambda}\frac{\partial (U\lambda)}{\partial \lambda} + \frac{\partial W}{\partial Z} = 0\,, \qquad (3)$$

where U and W are the velocities in the radial (λ) and axial (Z) direc-- tions, P the pressure and T the time .

Using slow flow conditions ($\nabla^2 P = 0$), (2) becomes

$$\left[\frac{\partial^2}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial}{\partial \lambda} + \frac{\partial^2}{\partial Z^2} \right]^2 W$$

$$= \frac{\partial}{\partial T} \left[\frac{\partial^2}{\partial \lambda^2} + \frac{1}{\lambda} \frac{\partial}{\partial \lambda} + \frac{\partial^2}{\partial Z^2} \right] W . \tag{4}$$

The boundary conditions are

$$\left. \begin{array}{l} U = V_1 \, Cos \, nT, \; W = 0 \text{ at } \lambda = 1 + \epsilon \, N_1(Z), \; T > 0 \\ \text{and } U = V_2 \, Cos \, nT, \; W = 0 \text{ at } \lambda = \sigma + \epsilon \, N_2(Z), \; T > 0 \end{array} \right\}, \tag{5}$$

where $\epsilon << 1$ is the roughness parameter, $N_1(Z)$ and $N_2(Z)$ are arbitrary functions of Z and σ is the ratio of the radius of the outer cylinder to that of the inner cylinder . V_1 and V_2 are the known non-dimensional velocities of suction / injection such that $V_1 = \sigma V_2$.

Let

$$\left. \begin{array}{l} P(\lambda, Z, T) = P_0(Z, T) + P_1(\lambda, Z, T) \\ U(\lambda, Z, T) = U_0(\lambda, T) + U_1(\lambda, Z, T) \\ W(\lambda, Z, T) = W_0(\lambda, T) + W_1(\lambda, Z, T) \end{array} \right\}, \tag{6}$$

where P_1, U_1 and W_1 are the variations caused by the roughness and P_0, U_0 and W_0 are the known quantities for the porous coaxial cylinolers without roughness .

Assuming

$$-\frac{\partial P_0}{\partial Z} = K \, Gos \;\; \mathrm{nT} = Re \left[K e^{inT} \right],$$

$$W_0 = Re\,[\,f_1(\lambda)\, e^{inT}\,] \text{ and } U_0 = Re\,[\,f_2(\lambda)\, e^{inT}\,]$$

and solving the resulting equations under the boundary conditions

$$f_1(1) = 0 = f_2(\sigma) \text{ and } f_1(1) = V_1 \text{ and } f_2(\sigma) = V_2\,,$$

we have

$$W_0(\lambda,T) = Re\left[\ \frac{K}{4}\,\frac{\{(\sigma^2-1)\log\lambda-(\lambda^2-1)\log\sigma\}}{\log\sigma}\ e^{inT}\right] \quad (7)$$

and

$$U_0(\lambda,T) = Re\left[\ \left\{\frac{V_1(\sigma^2-\lambda^2)+\sigma V_2(\lambda^2-1)}{\lambda(\sigma^2-1)}\right\} e^{inT}\right]. \quad (8)$$

Using (6), (1) and (3) we have.

$$\frac{\partial U_1}{\partial T} = -\frac{\partial P_1}{\partial\lambda} + \frac{\partial^2 U_1}{\partial Z^2} - \frac{\partial^2 W_1}{\partial\lambda\partial Z} \quad (9)$$

Equation (4) now becomes

$$\left[\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial}{\partial\lambda} + \frac{\partial^2}{\partial Z^2}\right]^2 W_1 = \frac{\partial}{\partial T}\left(\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial}{\partial\lambda} + \frac{\partial^2}{\partial Z^2}\right) W_1. \quad (10)$$

The boundary conditions now, are

$$\left.\begin{array}{l} U_0 + U_1 = V_1 \text{ Cos } nT,\ W_1 = -W_0 \text{ at } \lambda = 1+\epsilon N_1(Z), T>0, Z>0 \\ U_0 + U_1 = V_2 \text{ Cos } nT,\ W_1 = -W_0 \text{ at } \lambda = \sigma+\epsilon N_2(Z), T>0, Z>0 \\ \text{and} \\ W_1 = 0 \text{ when } \{1+\epsilon N_1(Z)\} < \lambda < \{\sigma+\epsilon N_2(Z)\}, T<0, Z=0 \end{array}\right\} \quad (11)$$

3. METHOD OF SOLUTION

Using Fourier sine transform, the solution of the equation (10), gives

$$W_1(\lambda,Z,T) = Re\,[\,\sqrt{(2/\pi)}\int_0^\infty [\,A(\xi)\,I_0(\xi\lambda) + B(\xi)\,K_0(\xi\lambda)$$

$$+\, C(\xi)\,I_0(m\lambda) + D(\xi)\,K_0(m\lambda)\,]\,sin\,(\xi Z)e^{inT}\,d\xi\,]. \tag{12}$$

Using (12), equation (9) gives

$$U_1(\lambda,Z,T) = Re\,[\,-\sqrt{(2/\pi)}\int_0^\infty [\,A(\xi)\,I_1(\xi\lambda) - B(\xi)\,K_1(\xi\lambda)$$

$$+\,\frac{\xi}{m}\,C(\xi)\,I_1(m\lambda) - \frac{\xi}{m}\,D(\xi)\,K_1(m\lambda)\,]\,cos\,(\xi Z)\,e^{inT}\,d\xi\,]. \tag{13}$$

Using (12) and (13), pressure $P_1(\lambda,Z,T)$ is given by

$$P_1(\lambda,Z,T) = Re\,[\,\sqrt{(2/\pi)}\,in\int_0^\infty \frac{1}{\xi}\,[\,A(\xi)\,I_0(\xi\lambda) + B(\xi)\,K_0(\xi\lambda)\,]$$

$$\times\,Cos\,(\xi Z)\,e^{inT}\,d\xi\,] + C, \tag{14}$$

where $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$, and C are the constants of integration. I_0, K_0 I_1 and K_1 are modified Bessel functions of first and second kind of order zero and one.

We assume

$$A(\xi) = A_0(\xi) + \epsilon\,A_1(\xi) + \cdots, \tag{15}$$

and similar expressions for $B(\xi)$, $C(\xi)$ and $D(\xi)$.

Using equations (11) to (15) and Fourier sine and cosine integral theorems, the complete expressions for the axial velocity radial velocity and pressure are

$$W(\lambda,Z,T) = W_0(\lambda,T) + W_1(\lambda,Z,T)$$

$$= \frac{K}{4}\left[\frac{(\sigma^2-1)\log\lambda}{\log\sigma} - \lambda^2 + 1\right]\cos nT + \epsilon\sqrt{(2/\pi)}$$

$$Re\int_0^\infty \{\,[\,\triangle A_{11}(\xi)\,I_0(\xi\lambda)$$

$$+ \triangle A_{31}(\xi)\,I_0(\xi\lambda) - \triangle B_{12}(\xi)\,K_0(\xi\lambda)$$

$$- \triangle B_{32}(\xi)\,K_0(\xi\lambda) + \triangle C_{13}(\xi)\,I_0(m\lambda) + \triangle C_{33}(\xi)\,I_0(m\lambda)$$

$$- \triangle D_{14}(\xi)\,K_0(m\lambda) - \triangle D_{34}(\xi)\,K_0(m\lambda)\,]\,\bar{N}_1(\xi)$$

$$- [\,\triangle A_{21}(\xi)\,I_0(\xi\lambda) + \triangle A_{41}(\xi)\,I_0(\xi\lambda) - \triangle B_{22}(\xi)\,K_0(\xi\lambda)$$

$$- \triangle B_{42}(\xi)\,K_0(\xi\lambda) + \triangle C_{23}(\xi)\,I_0(m\lambda) + \triangle C_{43}(\xi)\,I_0(m\lambda)$$

$$- \triangle D_{24}(\xi)\,K_0(m\lambda) - \triangle D_{44}(\xi)\,K_0(m\lambda)\,]\,\bar{N}_2(\xi)\,\}$$

$$\times\ \sin(\xi Z)\,e^{inT}\,d\xi, \tag{16}$$

$$U(\lambda,Z,T) = U_0(\lambda,T) + U_1(\lambda,Z,T)$$

$$= \left[\frac{V_1(\sigma^2-\lambda^2) + \sigma V_2(\lambda^2-1)}{\lambda(\sigma^2-1)}\right]\cos nT$$

$$+ \epsilon\,Re\,[\,-\sqrt{(2/\pi)}\int_0^\infty \{\,[\,\triangle A_{11}(\xi)\,I_1(\xi\lambda) + \triangle A_{31}(\xi)\,I_1(\xi\lambda)$$

$$+ \triangle B_{12}(\xi)\,K_1(\xi\lambda) + \triangle B_{32}(\xi)\,K_1(\xi\lambda) + \xi/m\triangle C_{13}(\xi)\,I_1(m\lambda)$$

$$+ \xi/m\triangle C_{33}(\xi)\,I_1(m\lambda) + \xi/m\,D_{14}(\xi)\,K_1(m\lambda)$$

$$+ \xi/m\triangle D_{34}(\xi)\,K_1(m\lambda)\,]\,\bar{N}_1(\xi) - [\,\triangle A_{21}(\xi)\,I_1(\xi\lambda)$$

$$+ \triangle A_{41}(\xi) I_1(\xi\lambda) + \triangle B_{22}((\xi) K_1(\xi\lambda) + \triangle B_{42}(\xi) K_1(\xi\lambda)$$

$$+ \xi/m \triangle C_{23}(\xi) I_1(m\lambda) + \xi/m\, C_{43}(\xi) I_1(m\lambda)$$

$$+ \xi/m \triangle D_{24}(\xi) K_1(m\lambda) + \xi/m \triangle D_{44}(\xi) K_1(m\lambda)]\, \bar{N}_2(\xi) \}$$

$$\times Cos(\xi Z)\, e^{inT}\, d\xi\,] \qquad (17)$$

and

$$P(\lambda,Z,T) = P_0(Z,T) + P_1(\lambda,Z,T)$$

$$= C - KZ\, Cos\, nT + \epsilon \sqrt{(2/\pi)}\, Re\, [\int_0^\infty in\, 1/\xi\, [\{ \triangle A_{11}(\xi) I_0(\xi\lambda)$$

$$+ \triangle A_{31}(\xi) I_0(\xi\lambda) - \triangle B_{12}(\xi) K_0(\xi\lambda) - \triangle B_{32}(\xi) K_0(\xi\lambda) \}\, \bar{N}_1(\xi)$$

$$- \{ \triangle A_{21}(\xi) I_0(\xi\lambda) + \triangle A_{41}(\xi) I_0(\xi\lambda) - \triangle B_{22}(\xi) K_0(\zeta\lambda)$$

$$- \Delta\, B_{42}(\xi) K_0(\xi\lambda) \}\, \bar{N}_2(\xi)]\, Cos(\xi Z)\, e^{inT}\, d\xi\,], \qquad (18)$$

where C denotes a constant, and

$\triangle A_{11}(Z)$, $\triangle B_{12}(Z)$, $\Delta C_{13}(Z)$ and $\Delta D_{14}(Z)$ are respectively minors of $I_0(Z)$, $K_0(Z)$, (m') and $K_0(m')$ in Δ, each multiplied by X / Δ.

$\Delta A_{21}(Z)$, $\Delta B_{22}(Z)$, $\Delta C_{23}(Z)$ and $\Delta D_{24}(Z)$ are respectively the minors of $I_0(\sigma Z)$, $K_0(\sigma Z)$ $I_0(\sigma m')$ and $K_0(\sigma m')$ in Δ, each multiplied by Y / Δ,

$\Delta A_{31}(Z)$, $\Delta B_{32}(Z)$, $\Delta C_{33}(Z)$ and $\Delta D_{34}(Z)$ are respectively

$$-\triangle A_{21}(1/l) - \triangle A_{41}(1/l)\,]\, I_0(\lambda/l) - [\,\triangle B_{12}(1/l)$$
$$+ \triangle B_{32}(1/l) - \triangle B_{22}(1/l) - \triangle B_{42}(1/l)\,]\, K_0(\lambda/l)\,\}$$
$$\times \cos(Z/l)\, e^{inT}\,], \tag{23}$$

where

$$1/l' = (1/l^2 + i\,n)^{1/2}.$$

5. NUMERICAL DISCUSSION

Axial and radial velocity profiles have been drawn for $\varepsilon = 0.1$, $\sigma = 2$ at various sections of the annulus for different values of nT.

Fig. 1 shows axial velocity profiles at different sections of the annulus for $nT = 0$ and $nT = \pi/3$ for the injection / suction velocities $V_1/k = 1 : V_2/k = 1/2$ and $V_1/k = 2 : V_2/k = 1$ at the inner and outer cylinders respectively. For $nT = 0$ they are shown in upper half (Which will be the same in the lower half) and for $nT = \pi/3$ in the lower half (which will be lhe same in the upper half) of the annulus . It is noted that increase in the injection / suction velocities st the inner and outer cylinders respectively results in the increase of the axial velocity at the section $Z = \pi/2$ while decrease is observed at $Z = 3\pi/2$. At the scetion $Z = 0$ and π, the axial velocity remains unaffected by injection/ suction velocities . For the case $nT = 0$, the profiles are almost parabolic .

Fig . 2 depicts axial velocity profiles in a similar way as shown in Fig . 1 for suction / injection velocities $V_1/k = -1 : V_2/k = -(1/2)$ and $V_1/k = -2 : V_2/k = -1$ at the inner and outer cylinders respectively . The axial velocity decreases with increase in the suction / injection velocities at the section $Z = \pi/2$, the decrease being much

sharper near the inner cylinder, while the axial velocity increases at the section $Z = 3\pi / 2$. The profiles now are almost parabolic at the section $Z = 3\pi/2$.

Fig 3 (*a*), (*b*), (*c*) and (*d*) respectively represent the radial velocities at sections $Z = 0, \pi/2, \pi$ and $3\pi/2$ of the annulus for $V_1/k = 1 : V_2/k = 1/2$, $V_1/k = 2 : V_2/k = 1$, $V_1/k = -1 : V_2/k = -(1/2)$ and $V_1/k = -2 : V_2/k = -1$ and $nT = 0$.

At all the four sections there is a uniform decrease in the radial velocity from inner to the outer cylinder .

Increase in the value of the waviness parameter ε, increases the purturbed parts of the velocities and the pressure .

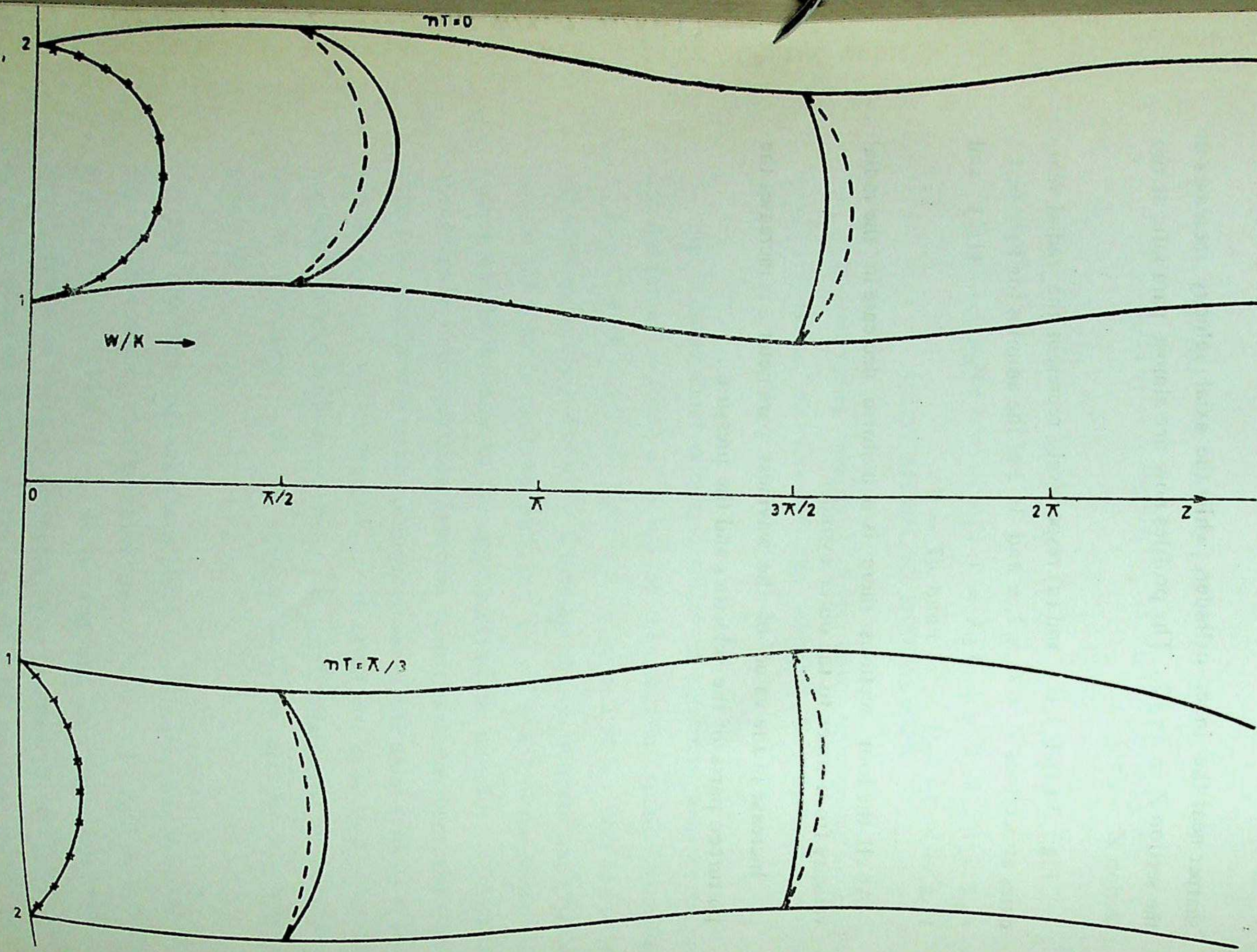

FIG 1 AXIAL VELOCITY PROFILES AT DIFFERENT SECTIONS OF THE ANNULUS FOR $\sigma = 2$, $\epsilon = 0{\cdot}1$ AND $V_1/K = 1$: $V_2/K = 1/2$ (----) AND $V_1/K = 2$: $V_2/K = 1$ (——).

FIG 1 AXIAL VELOCITY PROFILES AT DIFFERENT SECTIONS OF THE ANNULUS FOR $\sigma = 2$, $\epsilon = 0 \cdot 1$ AND $V_1/K = 1$: $V_2/K = 1/2$ (----) AND $V_1/K = 2$: $V_2/K = 1$ (——).

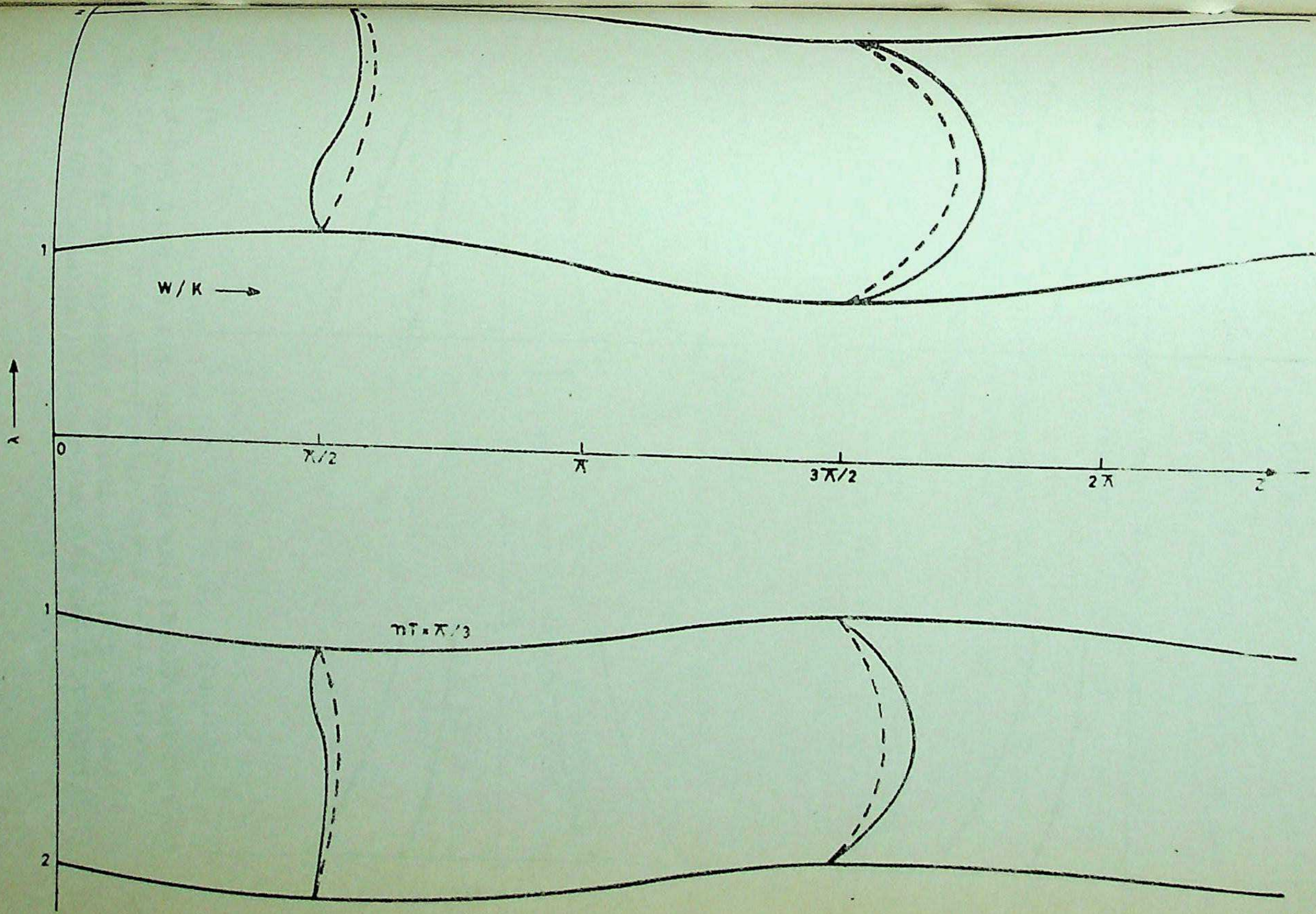

FIG 2 AXIAL VELOCITY PROFILES AT DIFFERENT SECTIONS OF THE ANNULUS FOR $\sigma = 2$, $\epsilon = 0 \cdot 1$ AND $V_1/K = -1$: $V_2/K = -1/2$ (-----) AND $V_1/K = -2$: $V_2/K = -1$ (——)

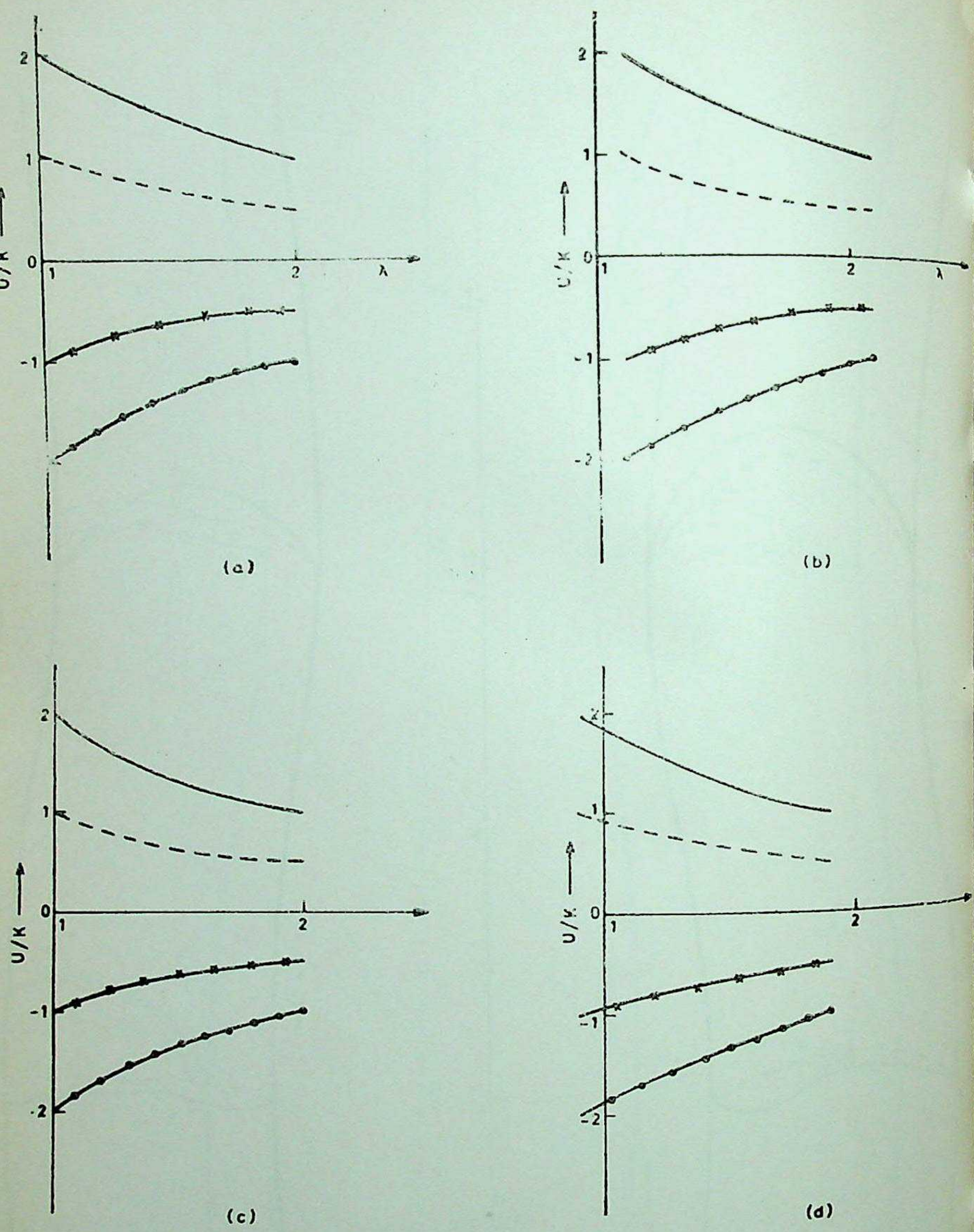

FIG. 3 RADIAL VELOCITY PROFILES FOR $\sigma = 2$, $\epsilon = 0{\cdot}1$, $nT = 0$ AND $V_1/K = 1$: $V_2/K = 1/2$ (-----), $V_1/K = 2$: $V_2/K = 1$ (———), $V_1/K = -1$: $V_2/K = -1/2$ (-x-x-), AND $V_1/K = -2$: $V_2/K = -1$ (-o-o-) AT DIFFERENT SECTIONS OF THE ANNULUS $Z = 0, \pi/2, \pi$ AND $3\pi/2$ RESPECTIVELY [(a), (b), (c) AND (d)]

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RANDOM VIBRATION OF A BEAM TO STOCHASTIC LOADING

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ABSTRACT

We study the random vibration of a beam of finite length under stochastic loading . The technique of the Fourier transform is used for the solution of the problem .

1. INTRODUCTION

In recent years, much attention has been devoted to the problems involving stochastic vibrations of beams . The motivation for such researches is due to various requirements such as wind effects and earthquake effects on structures, etc . Stochastic vibration of different types of beams has been studied by many authors (see [1], [3], [4], [5] and [6]) . In these problems, a mechanical system is excited by a random type load and the response of the system is of random nature. Its analysis is done with the help of stochastic process . The analytical tools for such study were borrowed srom the noise problems in communication theory and adapted to problems of structural dynamics([2],[3]). In dealing with stochastic processes, the averages very often used are the auto-correlation functions, and their Forier transforms are called the power spectral densities

In the present paper, we illustrate the Subject-matter of stochastic vibration by considering the vibrati ons of a beam of finite length, which is freely hinged at its ends, subjected to stochastic loading. The mean square values of the output displacement and the velocity are calculated with the help of auto-correlation function of applied load.

2, FORMULATION OF THE PROBLEM

Let a beam of finite length L, which is freely hinged at its ends. be subjected to a stochastic load

$$P(x,t) = \Psi(t, \delta(x - x'), (0 < x' < L)$$

where $\Psi(t)$ is stochastic in time t. If $y(x,t)$ is the transverse stochastic displacement of any point x of the beam at time t, then differential equation governing the motion of the beam is [7, p. 111, Eq. (50)]

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2} = \frac{P(x,t)}{EI}, \tag{1}$$

where $a^2 = EI/(\rho S)$, E is Young's modulus of the material forming the beam, I the moment of inertia of the cross-section of the beam with respect to line normal to both the x and y axes and passes through the centre of mass of cross-sectional area S and ρ is the mass per unit length of the beam.

The boundary conditions are [7, p. 116 Eq. (63)]

$$y = \frac{\partial^2 y}{\partial x^2} = 0 \text{ at } x = 0 \text{ and } x = L. \tag{2}$$

3. SOLUTION OF THE PROBLEM

Applying the Fourier transforms to (1), we have, finally the expression for the transverse displacement ([7], p. 118)

$$y(x,t) = \frac{2\,La}{\pi^2 E I} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) \frac{1}{n^2} \times$$

$$\times \int_0^t \Psi(\tau) \sin\left[\frac{n^2\pi^2 a\,(t-\tau)}{L^2}\right] d\tau \tag{3}$$

giving the deterministic solution of displacement in terms of some integrals .

The auto-correlation function for the output process $y(x,t)$ is defined as

$$R_y(x, \xi, t_1, t_2) = E\left[\, y(x, t_1)\, y(x, t_2) \,\right] \tag{4}$$

where E is the expectation .

Using (3) and (4) and putting $r = t_1 - \tau_1$, $s = t_2 - \tau_2$,

we get

$$R_y(x, \xi, t_1, t_2) = \left(\frac{2La}{\pi^2 E I}\right) \sum_{m,n=1}^{\infty} \frac{1}{n^2 m^2} \sin\left(\frac{n\pi x}{L}\right)$$

$$\sin\left(\frac{n\pi x'}{L}\right) \sin\left(\frac{m\pi\xi}{L}\right) \sin\left(\frac{m\pi\xi'}{L}\right)$$

$$\times \int_0^{t_1} \int_0^{t_2} \sin\left(\frac{n^2\pi^2 a}{L}\right) r \, \sin\left(\frac{m^2\pi^2 a}{L}\right) s$$

$$R_{\Psi}(t_1 - r, t_2 - s)\, dr\, ds \tag{5}$$

where the definition for R_{Ψ} is used .

We prescribe the load $\Psi(t)$, *to be* stationary stochastic process of known spectral density $S_{\Psi}(w)$ as [5] :

$$S_{\Psi}(\omega) = 2\pi S_0 \delta(\omega) \tag{6}$$

where S_0 is a real constant and $\delta(w)$ is Dirac s delta function .

Then, the auto-correlation function of $\Psi(t)$ is

$$R_{\Psi}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{\Psi}(\omega)\, e^{i\omega\tau}\, d\omega\,, \tag{6a}$$

from where, we get with the help of (6)

$$R_{\Psi}(\tau) = S_0\,. \tag{7}$$

Since the process $\Psi(t)$ is stationary, we have

$$R_{\Psi}(t_1 - r, t_2 - s) = R_{\Psi}(\tau) = S_0\,.$$

Using (7) in (5), we get

$$R_v(x, \xi, t_1, t_2) = y_0^2 \sum_{m,n=1}^{\infty} \frac{1}{n^4 m^4} \sin\frac{n\pi x}{L} \sin\frac{n\pi x'}{L}$$

$$\sin\frac{m\pi\xi}{L} \sin\frac{m\pi\xi'}{L} \times \left\{ 1 - \cos\frac{n^2\pi^2 a t_1}{L} \right\} \left\{ 1 - \cos\frac{m^2\pi^2 a t_2}{L} \right\} \tag{8}$$

giving auto-correlation function of the displacement ,

where $y_0^2 = 4L^4 So / \pi^2 E^2 I^2$. The series in (8) is convergent and the sum is finite for all values of t_1, t_2 . $R_y(x, \xi, t_1, t_2)$ varies directly as the constant S_0 corresponding to the spectral density of the stochastic process $\Psi(t)$. It varies inversely as the square of I .

In (8), putting $x = \xi$, $x' = \xi'$ and $t_1 = t_2 = t$, we have, the mean square value of the displacement

$$y_1^2 \ (x,t) - y_0^2 \sum_{m,n=1}^{\infty} \frac{1}{n^4 m^4} \quad sin \frac{n\pi x}{L} sin \frac{n\pi x'}{L} sin \frac{m\pi x}{L} sin \frac{m\pi x'}{L} \times$$

$$\times \left\{ 1 - cos \frac{n^2\pi^2 a^t}{L} \right\} \left\{ 1 - cos \frac{m^2\pi^2 at}{L} \right\} . \tag{9}$$

The velocity $V(x,t)$ is given by

$$V(x,t) = \frac{\partial y(x,t)}{\partial t} \quad . \tag{10a}$$

The auto-correlation function of the velocity is

$$R_v(x,\xi,t_1 t_2) = E \left[V(x,t_1) \, . \, V(x,t_2) \right]$$

$$= \frac{\partial^2 R_y(x,\xi,t_1,t_2)}{\partial t_1 \, \partial t_2} \, . \tag{10b}$$

Using (8), we have

$$R_v(x,\xi,t_1,t_2) = V_0^2 \sum_{m,n=1}^{\infty} \frac{1}{n^2 m^2} \ sin \frac{n\pi x}{L} \ sin \frac{n\pi x'}{L}$$

$$\sin \frac{m\pi\xi}{L} \sin \frac{m\pi\xi'}{L} \cdot \sin \frac{n^2\pi^2 at_1}{L} \sin \cdot \frac{m^2\pi^2 at_2}{L} \tag{11}$$

where $V_0^2 = 4\, L^2\, S_0\, a^2 \,/\, \pi^4\, E^2\, I^2$

Putting $x = \xi$, $x' = \xi'$ and $t_1 = t_2 = t$, we get the mean square velocity from (11) as

$$V_1^2\ (x, t) = V_0^2 \sum_{m,n=1}^{\infty} \frac{1}{n^2 m^2} \sin \frac{n\pi x}{L} \sin \frac{n\pi x'}{L} \sin \frac{m\pi x}{L} \sin \frac{m\pi x'}{L} \cdot$$

$$\cdot \sin \frac{n^2\pi_2 at}{L} \sin \frac{m^2\pi^2 at}{L}, \tag{12}$$

which obviously is zero at $t=0$.

4. NUMERICAL DISCUSSIONS AND GRAPHICAL REPRESENTATIONS

The calculations for mean square values of displacement (9) and velocity (12) are done for different values of non-dimensional time at/L at the point $x = x' = L/2$ of the beam . The calculated results are given in Table 1 . The tabulated results are plotted in Figures 1 and 2 .

It is seen from Figure 1 that mean square displacement is initially zero . It increases with time and attains maximum value after some time and then it gradually decreases to zero . It attains minima and maxima at equal intervals of time ,

Figure 2 represents mean square velocity . Starting from zero value it increases to maximum value and then decreases to zero value . This cyclee continues indefinitely .

TABLE 1

For $n = 1$ to 20, $m = 1$ to 20, $x = x' = L/2$

$at/L =$	0	1	2	3	4	5	6	7	8	9	10
$y_1^2/y_0^2 =$	0	3.6	0.15	1.5	1.4	0.2	3.6	0	3.6	0.14	1.5
$V_1^2/V_0^2 =$	0	0.25	0.63	0.9	1.0	0.6	0.3	0	0.2	0.6	0 9
at/L	–	11	12	13	14	15	16	17	18	19	20
y_1^2/y_0^2	–	1.4	1.2	2.5	0	3.6	0.12	1.7	1.4	0 2	3.5
V_1^2/V_0^2	–	1.0	0.6	0.3	0	0.2	0.6	0.9	1.0	0·7	0.3

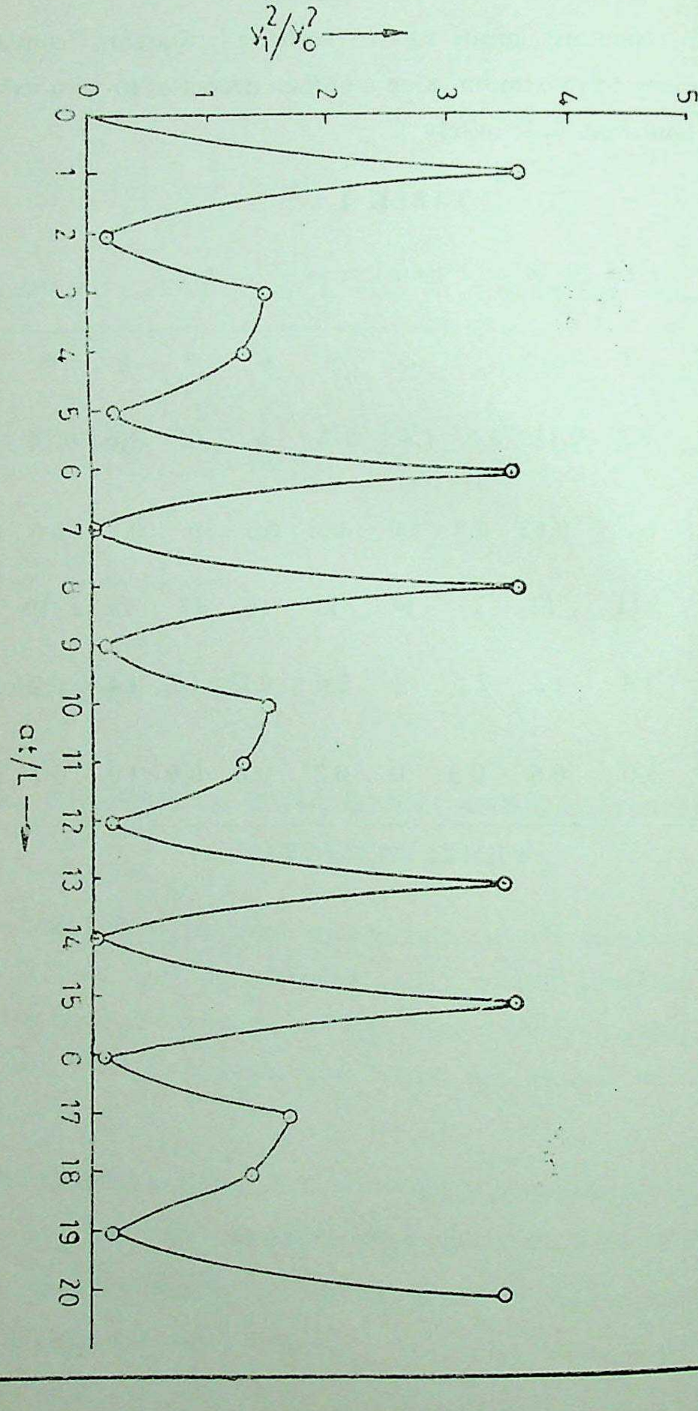

FIG.1.

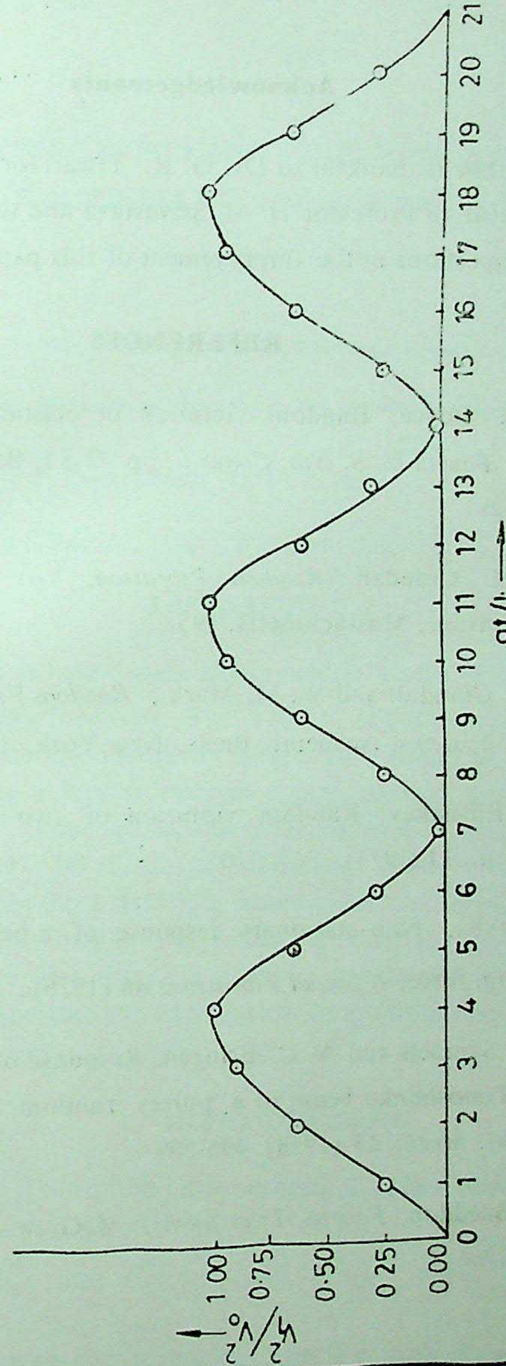
1.00
0.75
0.50
0.25
0.00
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
at/L

FIG. 2

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MULTIPLE HYPERGEOMETRIC FUNCTIONS RELATED TO LAURICELLA'S FUNCTIONS

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ABSTRACT

The papaer is based upon a commendable idea of interpolation between Lauricella's functions. The authors introduce three multiple hypergeometric functions related to Lauricella's functions and derive various generating relations, integral representations and recurrence realtions for them . Some special and confluent cases of these functions are also considered . Of course, as remarked at the end of Section 1 below, each of these three multiple hypergeometric functions as well as a fourth one (defined in the following paper by P. W. Karlsson [12]) is contained, as a special case, in the generalized Lauriclla hypergeometric function introduced , almost two decades ago, by H. M. Srivastava and M. C. Daoust [17] .

1. INTRODUCTION

Lauricella [13] introduced the multiple hypergeometric functions

$F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$.

Exton [7] introduced the following two multilple hypergeometric functions related to Lauricella's $F_D^{(n)}$:

(1.1) $\underset{(1)}{{}^{(k)}}\underset{(D)}{E^{(n)}}\ (a, b_1, \ldots, b_n\ ;\ c, c'\ ;\ x_1, \ldots, x_n)$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1 + \ldots + m_k} (c')_{m_{k+1} + \ldots + m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

(1.2) $\underset{(2)}{{}^{(k)}}\underset{(D)}{E^{(n)}}\ (a\ a', b_1, \ldots, b_n\ ;\ c\ ;\ x_1, \ldots, x_n)$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_k} (a')_{m_{k+1} + \ldots + m_n} (b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1 + \ldots + m_n}} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} .$$

Motivated by this work, Chandel [3] defined and studied the following function related to Lauricella's $F_C^{(n)}$:

(1.3) $\underset{(1)}{{}^{k}}\underset{C}{E^{(n)}}\ (a, a', b\ ;\ c_1, \ldots, c_n\ ;\ x_1, \ldots, x_n)$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_k} (a')_{m_{k+1} + \ldots m_n} (b)_{m_1 + \ldots + m_n}}{(c)_{m_1} \cdots (c_n)_{m_n}} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} .$$

In the same paper [3] , he also introduced certain other functions related to $E_A^{(n)}$, and $F_B^{(n)}$, but they are trivially reducible to combinations of Lauricell's functions . Since then we have been interested in finding new multiple hypergeometric functions related to $F_A^{(n)}$ and $F_B^{(n)}$. Fortunately, in the present paper . we are able to introduce the

following multiple hypergeometric functions related to Lauricella's functions [13] :

(1.4) $\quad {}^{(k)}F_{AC}^{(n)}\,(\,n,\,b,\,b_{k+1}\,,\ldots,\,b_n\,;\,c_1\,,\ldots,c_n\,;\,x_1\,,\ldots,\,x_n\,)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}\,(b)_{m_1+\ldots+m_k}\,(b_{k+1})_{m_{k+1}}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}} \times$$

$$\times\ \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\,.$$

For $k=0$, 1 it reduces to $F_A^{(n)}$, while for $k=n$ it reduces to $F_C^{(n)}$.

(1.5) $\quad {}^{(k)}F_{AD}^{(n)}\,(\,a,\,b_1\,,\ldots,\,b_n\,;\,c\,;\,c_{k+1}\,,\ldots,\,c_n\,;\,x_1\,,\ldots,\,x_n\,)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}\,(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_k}(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \times$$

$$\times\ \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

For $k=0$, I it reduces to $F_A^{(n)}$, while for $k=n$ it reduces to $F_D^{(n)}$,

(1.6) $\quad {}^{(k)}F_{BD}^{(n)}\,(\,a,\,a_{k+1}\,,\ldots,\,a_n,\,b_1\,,\ldots,\,b_n\,;\,c\,;\,x_1\,,\ldots,\,x_n\,)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_k}\,(a_{k+1})_{m_{k+1}}\ldots(a_n)_{m_n}\,(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}} \times$$

$$\times\ \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}\,.$$

For $k = 0, 1$, it reduces to $F_B^{(n)}$, and for $k = n$, it reduces to $F_D^{(n)}$. Some confluent forms special cases, generating relations, integral representations and recurrence relations have been discussed here.

It should be remarked here that, in the following paper, Karlsson [12] has defined one more multiple hypergeometric series of this class and has given the precise regions of convergence for all four of these multiple hypergeometric series. He has also presented several other properties of these functions, including, for example, Eulerian inetegral representations, transformations, and reduction formulas.

Just as (1.1), (1.2) and (1.3), the multiple hypergeometric functions defined by (1.4), (I.5) and (1.6) as well as the fourth one defined by Karlsson [12], are contained, as special cases, in the generalized Lauricella hypergeometric function of Srivastava and Daoust (*cf*. [17, p. 454] ; see also [11, pp 106–107] and [18, p. 37]).

2. CONFLUENT FORMS

For confluent forms, we consider

$$(2.1) \quad \lim_{b_{k+1},\ldots,b_n\to\infty} {}^{(k)}F_{AC}^{(n)}\left(a, b, b_{k+1},\ldots, b_n ; c_1,\ldots, c_n ; x_1,\ldots, x_k, \frac{x_{k+1}}{b_{k+1}},\ldots, \frac{x_n}{b_n}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_k}}{(c_1)_{m_1}\cdots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(1)}\phi_{AC}^{(n)}(a, b ; c_1,\ldots, c_n ; x_1,\ldots, x_n), \; k \neq n .$$

$$(2.2) \quad \lim_{b \to \infty} {}^{(k)}F_{AC}^{(n)} \left(a, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; \frac{x_1}{b}, \ldots, \frac{x_k}{b}, x_{k+1}, \ldots, x_n \right)$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_{k+1})_{m_{k+1}} \ldots (b_n)_{m_n}}{(c_1)_{m_1} \ldots (c_n)_{m_n}} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(2)}\phi_{AC}^{(n)} (a, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; x_1, \ldots, x_n) .$$

$$(2.3) \quad \lim_{c_{k+1}, \ldots, c_n \to 0} {}^{(k)}F_{AD}^{(n)} (a, b_1, \ldots, b_n ; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_k, c_{k+1} x_{k+1}, \ldots, c_n x_n)$$

$$= \sum_{m_1, \ldots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \ldots + m_n} (b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1 + \quad + m_k}} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(1)}\phi_{AD}^{(n)} (a, b_1, \ldots, b_n ; c ; x_1, \ldots, x_n), \; k \neq n .$$

$$(2.4) \quad \lim_{a_{k+1}, \ldots, a_n \to \infty} {}^{(k)}F_{BD}^{(n)} \left(a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n ; c ; x_1, \ldots, x_k, \frac{x_{k+1}}{a_{k+1}}, \ldots, \frac{x_n}{a_n} \right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_k}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdot+m_n}} \times$$

$$\times \quad \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{BD}\,(a, b_1,\dots, b_n \,;\, c \,;\, x_1,\dots, x_n),\; k \neq n$$

and

$$(2.5)\quad \lim_{a\to\infty}\ {}^{(k)}F^{(n)}_{BD}\,(a, a_{k+1},\dots, a_n, b_1,\dots, b_n \,;\, c \,;$$

$$\frac{x_1}{a},\dots,\frac{x_k}{a}, x_{k+1},\dots, x_n)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a_{k+1})_{m_{k+1}}\cdots(a_n)_{m_n}\,(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \times$$

$$\times \quad \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(1)}_{(2)}\phi^{(n)}_{BD}\,(a_{k+1},\dots, a_n, b_1,\dots, b_n \,;\, c \,;\, x_1,\dots, x_n),\; k \neq 0,$$

3. SPECIAL CASES

For $n = 4$ and $k = 3$, (1.4) gives

$$(4.1)\quad {}^{(3)}F^{(4)}_{AC}\,(a, b, b_4 \,;\, c_1, c_2, c_3, c_4 \,;\, x_1, x_2, x_3, x_4)$$

$$= K_2\,(a, a, a, a \,;\, b, b, b, b_4 \,;\, c_1, c_2, c_3, c_4 \,;\, x_1, x_2, x_3, x_4)$$

$$= {}^{(3)}_{(1)}E_C^{(4)} \,(b, b_4 \,;\, c, c_1 , c_2 , c_3 , c_4 \,;\, x_1 , x_2 , x_3 , x_4)$$

while; for $n = 4$ and $k = 2$, (1.4) reduces to

(3.2) $${}^{(2)}F_{AC}^{(4)} \,(a, b, b_3 , b_4 \,;\, c_1 , c_2 , c_3 , c_4 \,;\, x_1 , x_2 , x_3 , x_4)$$

$$= K_{10} \,(a, a, a, a;\, b, b, b, b_3 , b_4 \,;\, c_1 , c_2 , c_3 , c_4 \,;\, x_1, x_2, x_3, x_4).$$

For $n = 4$ and $k = 3$, (1 5) gives

(3.3) $${}^{(3)}F_{AD}^{(4)} \,(a, b_1, b_2 , b_3 , b_4 \,;\, c, c_4 \,;\, x_1 , x_2 , x_3 , x_4)$$

$$= K_{11} \,(a, a, a, a, b_1 , b_2 , b_3 , b_4 \,;\, c, c, c, c_4 \,;\, x_1\, x_2, x_3, x_4)$$

$$= {}^{(3)}_{(1)}E_D^{(4)} \,(a, b_1 , b_2 , b_3 , b_4 \,;\, c, c_4 \,;\, x_1 , x_2 , x_3 , x_4)$$

while, for $n = 4$ and $k = 2$, (1.5) gives

(3.4) $${}^{(2)}F_{AD}^{(4)} \,(a, b_1 , b_2 , b_3 , b_4 \,;\, c, c_3 , c_4 \,;\, x_1 , x_2 , x_3 , x_4)$$

$$= K_{13} \,(a, a, a, a \,;\, b_1 , b_2 , b_3 , b_4 \,;\, c, c, c_3 , c_4 \,;\, x_1 , x_2 , x_3 , x_4) .$$

Further , for $n = 4$ and $k = 3$, (1 6) yields

(3.5) $${}^{(3)}F_{BD}^{(4)} \,(a, a_4 , b_1 , b_2 , b_3 , b_4 \,;\, c \,;\, x_1 , x_2 , x_3 , x_4)$$

$$= K_{15} \,(a, a, a, a_4 \,;\, b_1 , b_2 , b_3 , b_4 \,;\, c, c, c, c \,;\, x_1 , x_2 , x_3 , x_4)$$

and, for $n = 4$ and $k = 2$, (1.6) gives

(3.6) $${}^{(2)}F_{BD}^{(4)} \,(a, a_3 , a_4 , b_1 , b_2 , b_3 , b_4 \,;\, c \,;\, x_1 , x_2 , x_3 , x_4)$$

$$= K_{21}\ (a, a, a_3, a_4 ; b_1, b_2, b_3, b_4 ; c, c, c, c ; x_1, x_2, x_3, x_4),$$

where $K_2, K_{10}, K_{11}, K_{13}, K_{15}, K_{21}$ are the multiple hypergeometric functions of four variables studied by Exton ([6] , [8], [11]) ,

For $n = 3$ and $k = 2$, (1.4) reduces to

(3.7) $${}^{(2)}F^{(3)}_{AC}\ (a, b, b_3 ; c_1, c_2, c_3 ; x_1, x_2, x_3)$$

$$= F_E\ (a, a, a, b_3\ b, b ; c_3, c_1, c_2 ; x_3, x_1, x_2)$$

while, for $n = 3$ and $k = 2$, (1.5) gives

(3.8) $${}^{(2)}F^{(3)}_{AD}\ (a, b_1, b_2 . b_3 ; c, c_3 ; x_1, x_2, x_3)$$

$$= F_G\ (a, a, a, b_3, b_1, b_2 ; c_3, c, c ; x_3, x_1, x_2)$$

$$= {}^{(1)}_{(1)}E^{(3)}_D\ (a, b_3, b_1, b_2 ; c_3, c ; x_3, x_1, x_2)$$

Further, for $n = 3$ and $k = 2$, (1.6) reduces to

(3.9) $${}^{(2)}F^{(3)}_{BD}\ (a, a_3, b_1, b_2, b_3 ; c ; x_1, x_2, x_3)$$

$$= F_S\ (a_3, a, a, b_3, b_1, b_2 ; c, c, c ; x_3, x_1, x_2)$$

$$= {}^{(1)}_{(2)}E^{(3)}_D\ (a_3, a, b_3, b_1 . b_2 ; c ; x_3, x_1, x_2),$$

where F_E, F_G and F_S are Laur:celia's hypergeometric functions of three variables studied, in these notations. by Saran ([14] ; see also Srivastava and Karlsson [18, pp. 42-43]) .

Similarly, we can give special cases of confluent hypergeometric functions also .

4. Generating Relations

From definitions, we obtain

(4.1) $$(1-t)^{-a}\ {}^{(k)}F^{(n)}_{AC}\left(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; \frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!}\, t^r\ {}^{(k)}F^{(n)}_{AC}(a+r, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n, x_1, \ldots, x_n),$$

(4.2) $$(1-t)^{-a}\ {}^{(k)}F^{(n)}_{AD}\left(a, b_1, \ldots, b_n ; c, c_{k+1}, \ldots, c_n ; \frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t}\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{r!}\, t^{(r)}\ {}^{(k)}F^{(n)}_{AD}(a+r, b_1, \ldots, b_n ; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_n),$$

(4.3) $$(1-t)^{-a}\ {}^{(k)}F^{(n)}_{BD}\left(a, a_{k+1}, \ldots, a_n, b_1, \ldots b_n ; c ; \frac{x_1}{1-t}, \ldots, \frac{x_k}{1-t}, x_{k+1}, \ldots, x_k\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ {}^{(k)}F^{(n)}_{BD}(a+r, a_{k+1}, \ldots, a_n, b_1, \ldots b_n ; c ; x_1, \ldots, x_n),$$

and

(4.4) $$(1-t)^{-b}\ {}^{(k)}F^{(n)}_{AC}\left(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; \frac{x_1}{1-t}, \ldots, \frac{x_k}{1-t}, x_{k+1}, \ldots, x_n\right)$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r}{r!}\ {}^{(k)}F^{(n)}_{AC}(a, b+r, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; x_1, \ldots, x_n).$$

Similarly, we can obtain generating relations for the confluent hypergeometric functions.

5· Integral Representations

Making an appeal to [16, p 101]

$$(5.1)\quad (\lambda,m) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-t}\, t^{\lambda+m-1}\, dt \ ,$$

and [5, p. 13]

$$(5.2)\quad \frac{1}{\Gamma(\lambda+m)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{t}\, t^{-\lambda-m}\, dt$$

$Re\ (\lambda) > 0,\ m = 0, 1,2\ ,\ldots,$ we derive the following integral representations :

$$(5.3)\quad {}^{(k)}F^{(n)}_{AC}\ (a, b, b_{k+1}\ , \ldots , b_n\ ;\ c_1\ , \ldots , c_n\ ;\ x_1\ , \ldots , x_n\ )$$

$$= \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^{\infty} \int_0^{\infty} e^{-(s+t)}\, s^{a-1}\, t^{b-1}\ {}_0F_1\ (- \ ;\ c_1\ ;\ x\ s\ t\ ) \ldots$$

$${}_0F_1\ (- \ ;\ c_k\ ;\ x_k\ s\ t\ )\ {}_1F_1\ (b_{k+1}\ ;\ c_{k+1}\ ;\ x_{k+1}\ s\ ) \ \ldots$$

$${}_1F_1\ (b_n\ ;\ c_n\ ;\ x_n\ s\ )\ ds\, dt\ ,$$

$$(5.4)\quad {}^{(k)}F^{(n)}_{AD}\ (a, b_1\ ,\ldots, b_n\ ;\ c,\ c_{k+1}\ ,\ldots, c_n\ ;\ x_1\ ,\ldots , x_n\ )$$

$$= \frac{\Gamma(c)}{2\pi i\,\Gamma(a)} \int_0^{\infty} \int_{-\infty}^{(0+)} e^{-s+t}\, s^{a-1}\, t^{-c}\ {}_1F_0\ \left(b_1\ ;-;\ \frac{x_1 s}{t}\ \right)$$

$$\ldots\ {}_1F_0\ \left(b_k\ ;-;\ \frac{x_k s}{t}\ \right)\ {}_1F_1\ (b_{k+1}\ ;\ c_{k+1}\ ;\ x_{k-1}\ s\ )$$

$$\ldots\ {}_1F_1\ (b_n\ ;\ c_n\ ;\ x_n\ s\ )\ ds\, dt\ ,$$

where $b_1, \ldots, b_k$ are negative integers, and

$$(5.5) \quad \overset{(k)}{F}\overset{(n)}{}_{BD} (a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n ; c ; x_1, \ldots, x_n)$$

$$= \frac{1}{\Gamma(a)\,\Gamma(a_{k+1}) \ldots \Gamma(a_n)\,\Gamma(b_1) \ldots \Gamma(b_n)} \int_0^\infty \ldots (2n-k+1) \ldots$$

$$\ldots \int_0^\infty e^{-(s + s_{k+1} + \cdots + s_n + t_1 + \cdots + t_n)} s^{a-1} s_{k+1}^{a_{k+1}-1} \ldots$$

$$s_n^{a_n-1} \quad t_1^{b_1-1} \ldots t_n^{b_n-1}$$

$${}_0F_1 [-; c; x_1 t_1 s + \ldots + x_k t_k s + x_{k+1} s_{k+1} t_{k+1}$$

$$+ \ldots + x_n s_n t_n] \, ds \, ds_{k+1} \ldots ds_n \, dt_1 \ldots dt_n ,$$

$$Re(a), Re(a_{k+1}), \ldots, Re(a_n), Re(b_1), \ldots, Re(b_n) > 0 .$$

Similarly, we can obtain integral representations for the confluent forms .

6. Recurrence Relations

For further investigations, we shall use (Exton ([10, p. 115 (3.5)])

$$(6.1) \quad {}_0F_1(-; c-1; x) - {}_0F_1(-; c; x) - \frac{x}{c(c-1)} {}_0F_1(-; c+1; x) = 0 ,$$

and (Slater [15, p. 19])

$$(6.2) \quad c\, {}_1F_1(a; c; x) - c\, {}_1F_1(a-1; c; x) - x\, {}_1F_1(a; c+1, x) = 0,$$

$$(6.3) \quad (1+a-c)\, {}_1F_1(a; c; x) - a\, {}_1F_1(a+1; c; x) + (c-1)\, {}_1F_1(a; c-1; x) = 0 .$$

By an appeal to (6.1), (6.2) and (6.3), the relation (5.3) gives the following recurrence relations :

(6 4) $$^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; x_1, \ldots, x_n)$$

$$= {}^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_{j-1}, c_j-1, c_{j+1}, \ldots, c_n ; x_1, \ldots, x_n)$$

$$- \frac{x_j a b}{c_j (c_j-1)} {}^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_{j-1}, c_j+1,$$

$$c_{j+1}, \ldots, c_n ; x_1, \ldots, x_n),$$

where $j = 1, \ldots, k$;

(6.5) $$^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; x_1, \ldots, x_n)$$

$$= {}^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_{j-1}, b_j-1, b_{j+1}, \ldots, b_n ;$$

$$c_1, \ldots, c_n ; x_1, \ , x_n)$$

$$- \frac{a x_j}{c_j} {}^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_{j-1}, c_j+1,$$

$$c_{j+1}, \ldots, c_n ; x_1, \ldots, x_n),$$

where $j = k+1, \ldots, n$;

(6.6) $$(1+b_j-c_j) {}^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_n ; x_1, \ldots, x_n)$$

$$= b_j {}^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_{j-1}, b_j+1, b_{j+1}, \ldots, b_n ;$$

$$c_1, \ldots, c_n ; x_1, \ldots, x_n)$$

$$- (c_j-1) {}^{(k)}F^{(n)}_{AC}(a, b, b_{k+1}, \ldots, b_n ; c_1, \ldots, c_k, \ldots, c_{j-1},$$

$$c_j-1, c_{j+1}, \ldots, c_n ; x_1, \ldots, x_n),$$

where $j = k + 1, \ldots, n$.

Similarly, from (5.4) and (5.5) we derive

$$(6.7) \quad {}^{(k)}F_{AD}^{(n)}(a, b_1, \ldots, b_n ; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_n)$$

$$= {}^{(k)}F_{AD}^{(n)}(a, b_1, \ldots, b_k, b_{k+1}, \ldots, b_{j-1}, b_j - 1, b_{j+1}, \ldots, b_n ; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_n)$$

$$- \frac{x_j a}{c_j} \, {}^{(k)}F_{AD}^{(n)}(a, b_1, \ldots, b_n ; c, c_{k+1}, \ldots, c_{j-1}, c_j + 1, c_{j+1}, \ldots, c_n ; x_1, \ldots, x_n),$$

where $j = k + 1, \ldots, n$, and

$$(6.8) \quad {}^{(k)}F_{BD}^{(n)}(a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n ; c ; x_1, \ldots x_n)$$

$$= {}^{(k)}F_{BD}^{(n)}(a, a_{k+1}, \ldots, a_n, b_1, \ldots, b_n ; c - 1 ; x_1, \ldots, x_n)$$

$$- \frac{a}{c(c-1)} \left[x_1 b_1 \, {}^{(k)}F_{BD}^{(n)}(a + 1, a_{k+1}, \ldots, a_n, b_1 + 1, \ldots, b_n ; c + 1 ; x_1, \ldots, x_n) \right.$$

$$\left. + \ldots + x_k b_k \, {}^{(k)}F_{BD}^{(n)}(a + 1, a_{k+1}, \ldots, a_n, b_1, \ldots, b_{k-1}, b_k + 1, b_{k+1}, \ldots, b_n ; c + 1 ; x_1, \ldots, x_n) \right]$$

$$- \frac{1}{c(c-1)} \left[x_{k+1} a_{k+1} b_{k+1} \, {}^{(k)}F_{BD}^{(n)}(a, a_{k+1} + 1, a_{k+2}, \ldots, a_n, b_1, \ldots, b_k, b_{k+1} + 1, \ldots, b_n ; c + 1 ; x_1, \ldots, x_n) \right.$$

$$+ \dots + x_n a_n b_n{}^{(k)} F_{BD}^{(n)} (a, a_{k+1}, \dots, a_{n-1}, a_n + 1,$$

$$b_1, \dots, b_k, b_{k+1}, \dots, b_{n-1}, b_n + 1 ; c + 1 ; x_1, \dots, x_n)]$$

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ON INTERMEDIATE LAURICELLA FUNCTIONS

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ABSTRACT

For the four intermediate Lauricella functions, of which three were defined in the preceding paper [1] and a fourth one is defined by the multiple series (1.1) below, we establish regions of convergence together with some integral representations, transformations, and reduction formulas. Of course, as remarked in [1] as well as at the end of Section 1 below, each of these intermediate Lauricella functions is contained in the generalized Lauricella function of several variables, which was introduced earlier by H. M Srivastava and M C Daoust [6].

1. INTRODUCTION

With the four Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$, $F_D^{(n)}$ in mind, one might ask whether functions could be constructed that are (in a suitable sense) intermediate between pairs of Lauricella functions. Clearly, there are $\binom{4}{2}=6$ cases to be considered ; and, in the preceding paper, Chandel and Gupta [1] in fact introduced three intermediate functions ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(k)}$, and ${}^{(k)}E_{BD}^{(n)}$, see [1, Eqs. (1.4), (1.5), (1.6)]. An attempt to construct AB-and BC-functions on similar lines leads

to products. In the sixth case, however, we obtain a new function defined by the series

$$(1.1)\quad \overset{(k)}{F}{}^{(n)}_{CD}\,[\,a, b, b_1, \dots, b_k\,;\, c, c_{k+1}, \dots, c_n\,;\, x_1, \dots, x_n\,] =$$

$$= \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b)_{m_{k+1}+\dots+m_n} (b_1)_{m_1} \cdots (b_k)_{m_k}}{(c)_{m_1+\dots+m_k} (c_{k+1})_{m_{k+1}} \cdots (c_n)_{m_n}} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n} .$$

It reduces to $F_C^{(n)}$ for $k = 0$, and to $F_D^{(n)}$ for $k = n$.

In the present paper we shall supplement the investigation in [1] by establishing regions of convergence for the four functions together with some integral representations, transformations, and reduction formulas. The four functions are, incidentally, contained in the wide class of multiple hypergeometric functions introdnced by Srivastava and Daoust ([6, p 454 *et seq.*] ; see also [7, p 37]).

2. Regions of convergence

It is a well-knewn consequence of Stirling's theorem that the region of convergence D in absolute space for a hypergeometric series is independent of the parameters. (Exceptional values for which the series becomes meaningless or terminating are tacitly excluded.) If we choose suitable parameter values and take absolute values of all terms, a simpler series σ with region of convergence D emerges; and in many cases D is fairly easily obtained from σ. (Some examples of the method may be found in [2], [3], and [4].) Frequently,

this approach is superior to the application of Horn's theorem, especially because the latter grows complicated as the number of variables increases, compare [7, Chapter 4 and Section 9.2].

In the present case, the following results were found .

Series	Region of convergence
$\overset{(k)\ (n)}{F}_{AC}$	$(\sqrt{\lvert x_1 \rvert} + \ldots + \sqrt{\lvert x_k \rvert})^2 + \lvert x_{k+1} \rvert + \ldots + \lvert x_n \rvert < 1$
$\overset{(k)\ (n)}{F}_{AD}$	$max\,(\lvert x_1 \rvert, \ldots, \lvert x_k \rvert) + \lvert x_{k-1} \rvert + \ldots + \lvert x_n \rvert < 1$
$\overset{(k)\ (n)}{F}_{BD}$	$max\,(\lvert x_1 \rvert, \ldots, \lvert x_n \rvert) < 1$
$\overset{(k)\ (n)}{F}_{CD}$	$max\,(\lvert x_1 \rvert, \ldots, \lvert x_k \rvert) + (\sqrt{\lvert x_{k+1} \rvert} + \ldots + \sqrt{\lvert x_n \rvert})^2 < 1$

Proofs are obtained by taking all parameters equal to unity. To facilitate printing, we shall let

$$\underset{(k)}{\Sigma} \quad \text{and} \quad \underset{(n-k)}{\Sigma}$$

denote summations over $m_1, \ldots, m_k$ and $m_{k+1}, \ldots, m_n$, respectively, all indices run from 0 to ∞ . (The trivial cases $k = 0$ and $k = n$ are excluded .) For brevity we introduce

$$(2{\cdot}1) \quad \begin{cases} K = m_1 + \ldots + m_k, \qquad N = m_{k+1} + \ldots + m_n\,, \\ Y = max\,(\lvert x_1 \rvert, \ldots, \lvert x_k \rvert)\,, Z = \lvert x_{k+1} \rvert + \ldots + \lvert x_n \rvert\,. \end{cases}$$

In conjunction with $\overset{(k)\ (n)}{F_{AC}}$ we now consider the series

$$\sigma_1 = \sum_{(k)} \sum_{(n-k)} \frac{(K+N)!\,K!\,|x_1|^{m_1} \ldots |x_n|^{m_n}}{(m_1! \ldots m_k!)^2\, m_{k+1}! \ldots m_n!} =$$

$$= \sum_{(k)} \left[\frac{K!}{m_1! \ldots m_k!} \right]^2 |x_1|^{m_1} \ldots |x_k|^{m_k}$$

$$\sum_{(n-k)} \frac{(1+K)_N\, |x_{k+1}|^{m_{k+1}} \ldots |x_n|^{m_n}}{m_{k+1}! \ldots m_n!} .$$

The inner series equals $(1-Z)^{-1-K}$. Hence ,

$$\sigma_1 = (1-Z)^{-1} \overset{(k)}{F_C} \left[1, 1; 1, \ldots, 1 ; \frac{|x_1|}{1-Z}, \ldots, \frac{|x_k|}{1-Z} \right] .$$

The region of convergence is thus given by $Z < 1$ and

$$\sqrt{(|x_1|/(1-Z)} + \ldots + \sqrt{(|x_k|)/(1-Z))} < 1 .$$

A little rearrangement now yields the assertion for $\overset{(k)\ (n)}{F_{AC}}$. For the series

For the series $\overset{(k)\ (n)}{F_{AD}}$ we proceed as follows ,

$$\sigma_2 = \sum_{(k)} \sum_{(n-k)} \frac{(K+N)!\,|x_1|^{m_1} \ldots |x_n|^{m_n}}{K!\, m_{k+1}! \ldots m_n!} =$$

$$= \sum_{(k)} |x_1|^{m_1} \ldots |x_k|^{m_k} \sum_{(n-k)} \frac{(1+K)_N\, |x_{k+1}|^{m_{k+1}} \ldots |x_n|^{m_n}}{m_{k+1}! \ldots m_n!} .$$

Again, the inner series equals $(1-Z)^{-1-K}$, and we find

$$\sigma_2 = (1-Z)^{-1} \prod_{j=1}^{k} \sum_{m_j=0}^{\infty} \left[\frac{|x_j|}{1-Z} \right]^{m_j}.$$

Thus we must have $Z < 1$ and $|x_j| < 1-Z, j \in \{1,\dots,k\}$, or

$Y+Z < 1$, which is the assertion for ${}^{(k)}F^{(n)}_{AD}$.

In the third case we consider the series

$$\sigma_3 = \sum_{(k)} \sum_{(n-k)} \frac{K!\, m_{k+1}! \dots m_n!}{(K+N)!} |x_1|^{m_1} \dots |x_n|^{m_n}.$$

Clearly,

$$\sigma_3 < \sum_{(k)} \sum_{(n-k)} |x_1|^{m_1} \dots |x_n|^{m_n},$$

which implies convergence of ${}^{(k)}F^{(n)}_{BD}$ if $|x_1|,\dots,|x_n|$ are all below unity. Ou the other hand, upon discarding terms we obtain

$$\sigma_3 > \sum_{m_j=0}^{\infty} |x_j|^{m_j}, \qquad j \in \{1,\dots,n\},$$

which means that ${}^{(k)}F^{(n)}_{BD}$ is divergent if the modulus of any variable exceeds unity. This completes the third case.

Finally, for ${}^{(k)}F^{(n)}_{CD}$ we consider

$$\sigma_4 = \sum_{(n-k)} \left[\frac{N!}{m_{k+1}! \dots m_n!} \right]^2 |x_{k+1}|^{m_{k+1}} \dots |x_n|^{m_n}\ \sigma_5,$$

where

$$\sigma_5 = \sum_{(k)} \frac{(1+N)_K}{K\,!} \; | x_1 |^{m_1} \ldots | x_n |^{m_n} =$$

$$= \sum_{p=0}^{\infty} \frac{(1+N)_p}{p\,!} \sum_{k=p} | x_1 |^{m_1} \ldots | x_k |^{m_k} .$$

Clearly, one of the terms in the inner series is Y^p. Thus,

$$\sigma_5 > \sum_{p=0}^{\infty} \frac{(1+N)_p \, Y^p}{p\,!} = (1-Y)^{-1-N} ,$$

which implies

$$\sigma_4 > (1-Y)^{-1} F_C^{(n-k)} \left[1, 1;\; 1 ,\ldots, 1 \;;\; \frac{| x_{k+1} |}{1-Y} \;,\; ,\; \frac{| x_n |}{1-Y} \right]$$

On the other hand, the number of terms in the sum with $K=p$ does not exceed $(p+1)^k$, and for any $\varepsilon > 0$ we can find an $A > 0$ such that $(p+1)^k < A(1+\varepsilon)^p$ for all $p > 0$. Consequently,

$$\sigma_5 < \sum_{p=0}^{\infty} \frac{(1+N)_p}{p\,!} \; A\,(1+\varepsilon)^p Y^p = A\,(\,1-(1+\varepsilon)\,Y\,)^{-1-N}$$

and

$$\sigma_4 < A\,(1-(1+\varepsilon)Y)^{-1} F_C^{(n-k)} \left[1 , 1 ; 1 ,\ldots, 1 \;;\; \frac{| x_{k+1} |}{1-(1+\varepsilon)Y} ,\ldots, \frac{| x_n |}{1-(1+\varepsilon)Y} \right].$$

From the inequalities for σ_4 it follows, since ε may be arbitrarily small, that the region of convergence is given by

$$\sqrt{(\, | x_{k+1} | / (1-Y)\,)} + \cdots + \sqrt{(\, | x_n | / (1-Y)\,)} < 1 .$$

The fourth assertion now follows.

3. Integral representations of Euler's type

Since there is no simple Eulerian integral for $F_C^{(n)}$ we can not expect such representations to exist for ${}^{(k)}F_{AC}^{(n)}$ and ${}^{(k)}F_{CD}^{(n)}$. In the other cases, however, we have

$$(3.1)\quad {}^{(k)}F_{BD}^{(n)}\,[\,a,\,a_{k+1},\dots,a_n,\,b_1,\dots,b_n\,;\,c\,;\,x_1,\dots,x_n\,] =$$

$$= \frac{\Gamma(c)}{\Gamma(b_1)\dots\Gamma(b_n)\,\Gamma(c-b_1-\dots-b_n)} \times$$

$$\times \int_{T_n} u_1^{b_1-1}\dots u_n^{b_n-1}(1-u_1-\dots-u_n)^{c-b_1-\dots-b_n-1}\times$$

$$\times\ (1-u_{k+1}x_{k+1})^{-a_{k+1}}\dots(1-u_n x_n)^{-a_n}\ \times$$

$$\times\ (1-u_1x_1-\dots u_kx_k)^{-a}\,du_1\dots du_n\,,$$

where

$$T_n = \{\,(u_1,\ ,u_n)\mid 0\leqslant u_1\wedge\dots\wedge 0\leqslant u_n\wedge u_1+\dots+u_n\leqslant 1\,\},$$

$$Re\ c > Re\,(\,b_1+\dots+b_n\,)\,,\ Re\ b_j>0\,,\ j\in\{1,\dots,n\}\,;$$

and

$$(3.2)\quad {}^{(k)}F_{AD}^{(n)}\,[\,a,\,b_1,\dots,b_n\,;\,c\,,\,c_{k+1},\dots,c_n\,;\,x_1,\dots,x_n\,] =$$

$$= \frac{\Gamma(c)\,\Gamma(c_{k+1})\dots\Gamma(c_n)}{\Gamma(b_1)\dots\Gamma(b_n)\,\Gamma(c-b_1-\dots-b_k)\,\Gamma(c_{k+1}-b_{k+1})\dots\Gamma(c_n-b_n)}\times$$

$$\times \int_{S_{n,k}} u_1^{b_1-1} \ldots u_n^{b_n-1} (1-u_1 - \ldots - u_k)^{c-b_1 - \ldots - b_k-1}$$

$$\times (1-u_{k+1})^{c_{k+1}-b_{k+1}-1} \ldots (1-u_n)^{c_n-b_n-1} \times$$

$$\times (1-u_1x_1 - \ldots - u_nx_n)^{-a} \, du_1 \ldots du_n ,$$

where

$$S_{n,k} = \{ (u_1 ,\ldots, u_n) \mid 0 \leqslant u_1 \wedge \ldots \wedge 0 \leqslant u_k \wedge u_1 + \ldots + u_k \leqslant 1 \wedge$$

$$\wedge \; 0 \leqslant u_{k+1} \leqslant 1 \wedge \ldots \wedge 0 \leqslant u_n \leqslant 1 \} ,$$

$Re\, c > Re\, (b_1 + \ldots + b_k)$, $Re\, b_j > 0$, $j \in \{ 1 ,\ldots, k \}$,

$Re\, c_i > Re\, b_i > 0$, $i \in \{ k + 1 ,\ldots, n \}$.

The integral representations are proved in the usual way by expansion of factors containing $x_1 ,\ldots, x_n$. Incidentally, particular cases ($n = 3$, $k = 2$) of both representations were given by Saran [5, Eqs (3.10) and (3.5)].

4. Transformations

The domain of integration in (2.2) and the form of the integrand are both preserved under the transformations

$$t_1 = 1 - u_1 - \ldots - u_k ,\; t_2 = u_2 ,\ldots, t_n = u_n \quad (k \geqslant 1) ,$$

and

$$t_1 = u_1 ,\ldots, t_{n-1} = u_{n-1} ,\; t_n = 1 - u_n \qquad (k \leqslant n-1) .$$

The resulting transformations of Euler's type read ,

$$(4.1) \qquad \overset{(k)}{F}{}^{(n)}_{AD} [a, b_1 ,\ldots, b_n ; c, c_{k+1} ,\ldots, c_n ; x_1 ,\ldots, x_n] (1-x_1)^{a} =$$

$$= \overset{(k)}{F}{}^{(n)}_{AD} \, [\, a,\ c-(b_1+\ \dots\ +b_k)\ ,\ b_2\ ,\dots,\ b_n\ ;\ c,\ c_{k+1}\ ,\dots,\ c_n\ ;$$

$$\frac{-x_1}{1-x_1}\ ,\ \frac{x_2-x_1}{1-x_1}\ ,\dots,\ \frac{x_k-x_1}{1-x_1}\ ,\ \frac{x_{k+1}}{1-x_1}\ ,\dots,\ \frac{x_n}{1-x_1}\ \Bigg]\ ,$$

and

$$(4.2)\qquad \overset{(k)}{F}{}^{(n)}_{AD} \, [\, a,\ b_1\ ,\dots,\ b_n\ ;\ c,\ c_{k+1}\ ,\dots,\ c_n\ ;\ x_1\ ,\dots,\ x_n\ ]\ (1-x_n)^a =$$

$$= \overset{(k)}{F}{}^{(n)}_{AD} \, [\, a\ ,\ b_1\ \dots\ ,\ b_{n-1}\ ,\ c_n-b_n\ ;\ c\ ,\ c_{k+1}\ ,\dots,\ c_n\ ;$$

$$\frac{x_1}{1-x_n}\ ,\dots,\ \frac{x_{n-1}}{1-x_n}\ ,\ \frac{-x_n}{1-x_n}\ \Bigg]\ .$$

These are , however only two of several transformations which can be obtained in a similar way . Particular cases ($n = 3$. $k = 2$) are due to Saran [5 , Eqs. (5.3) and (5.2)] .

5. Reducible cases

Except for the function $\overset{(k)}{F}{}^{(n)}_{AC}$ there is a reduction formula in case the first k variables are equal . The formulas read ,

$$(5.1)\qquad \overset{(k)}{F}{}^{(n)}_{AD} \, [\, a\ ,\ b_1\ ,\dots,\ b_n\ ;\ c\ ,\ c_{k+1}\ ,\dots,\ c_n\ ;\ x\ ,\dots,\ x\ ,\ x_{k+1}\ ,\dots,\ x_n\ ] =$$

$$= F_A^{(n-k+1)}\, [\, a\ ,\ b_1\ +\ \dots\ +\ b_k\ ,\ b_{k+1}\ ,\dots,\ b_n\ ;\ c\ ,\ c_{k+1}\ ,\dots,\ c_n\ ;$$

$$x,\ x_{k+1}\ ,\dots,\ x_n\ ]\ ,$$

(5 2) $${}^{(k)}F^{(n)}_{BD}\,[\,a\,,\,a_{k+1}\,,\dots,\,a_n\,,\,b_1\,,\dots,\,b_n\,;\,c\,;\,x\,,\dots,\,x\,,\,x_{k+1}\,,\dots,\,x_n\,] =$$

$$= F_B^{(n-k+1)}\,[\,a\,,\,a_{k+1}\,\dots,\,a_n\,,\,b_1\,+\dots+\,b_k,\,b_{k+1}\,,\dots,\,b_n\,;\,c\,;$$

$$x\,,\,x_{k+1}\,,\dots,\,x_n\,]\,,$$

(5.3) $${}^{(k)}F^{(n)}_{CD}\,[\,a,\,b,\,b_1\,,\dots,\,b_k\,;\,c\,,\,c_{k+1}\,,\dots,\,c_n\,,\,x\,,\dots,\,x,\,x_{k+1}\,,\dots,\,x_n\,]=$$

$$= {}^{(1)}F^{(n-k+1)}_{CD}\,[\,a,\,b,\,b_1\,+\dots+\,b_k\,;\,c,\,c_{k+1}\,,\dots,\,c_n\,;\,x,\,x_{k+1}\,,\dots,\,x_n\,].$$

In all cases the function on the left-hand side is written as a series with summation indices $m_{k+1}\,,\dots,\,m_n$, such that the general term contains a $F_D^{(k)}$ with equal variables to which we apply Lauricella's reduction formula

(5.4) $$F_D^{(k)}\,[A\;\; b_1\,,\dots,\,b_k\,;\,C\,;\,x\,,\dots,\,x] = {}_2F_1\,[A,\,b_1+\dots+b_k\,;\,C\,;\,x]$$

Equality of parameters may give rise to reduction formulas , too . With

$$X = x_{k+1} + \dots + x_n$$

for brevity , we have

(5 5) $${}^{(k)}F^{(n)}_{AC}\,[\,a,\,b,\,c_{k+1}\,,\dots,\,c_n\,;\,c_1\,,\dots,\,c_n\,;\,x_1\,,\dots,\,x_n\,] =$$

$$= (1-X)^{-a}\;F_C^{(k)}\left[\,a,\,b;\,c_1\,,\dots,\,c_k\,;\,\frac{}{1-X}\,,\dots,\,\frac{x_k}{1-X}\right],$$

and

(5.6) $${}^{(k)}F^{(n)}_{AD}\,[\,a,\,b_1\,,\dots,\,b_n\,;\,c\,,\,b_{k+1}\,,\dots,\,b_n\,;\,x_1\,,\dots,\,x_n\,] =$$

$$= (1-X)^{-a} \quad F_D^{(k)}\left[a, b_1, \ldots, b_k \; ; \; c \; ; \; \frac{x_1}{1-X}, \ldots, \frac{x_k}{1-X} \right].$$

These are proved by the same series manipulations that we used for σ_1 and σ_2 .

Further reduction formulas could be obtained from the transformations mentioned in Section 4 .

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A PROPERTY OF CONTRACTIVE TYPE MAPPINGS IN 2-METRIC SPACES

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ABSTRACT

All the contractive type mappings in the theorems in [2] to [12] have property (H) in a 2-metric space . If a mapping f possesses property (H) in an Euclidean 2-metric space, then the points given by the iteration of f at any point in the Euclidean 2-metric space are collinear . Not all 2-metric spaces have property (H^*) .

1. INTRODUCTION

The concept of 2-metric space has been investigated by S. Gähler in [1] . Many contractive type principles on 2-metric spaces have been proved by K Iséki [2] , M. S Khan [3] , S. N. Lal and A. K. Singh ([4] , [5]) , S N. Lal and Mohan Das ([6] , [7]) , B. E. Rhoades [8] , A. K. Sharma [9], ([10], [11]), and S. L. Singh [12] The purpose of this article is to show that all the fixed point theorems in [2] to [12] can apply only to a self-mapping f, which is defined in a 2-metric space (X, d) and has the following property denoted by property (H) :

" $d\,(f^i x \,;\, f^j x \,,\, f^k x) = 0$, for all $i, j, k \in I^+$

and all $x \in X$, where $f^0 x = x$, $\forall x \in X$;

$f^1 x = fx$; $f^2 x = f^0 \ fx$; etc . ''

It follows that all the theorems in [2] to [12] cannot apply to a mapping which does not have the above property .

2. MAIN RESULTS (PART I)

From [1] we have :

Definition 1. A 2-metric space is a space X with a non-negative real-valued function d on $X \times X \times X$ satisfying the following conditions :

(i) to each pair of distinct points x, y in X, there exists a point z in X such that

$d(x, y, z) \neq 0$.

(ii) $d(x, y, z) = 0$, when at least two of x, y, z are equal .

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$.

(iv) $d(x, y, z) \leqslant d(u, y, z) + d(x, u, z) + d(x, y, u)$ (simplex inequality) .

Definition 2. Let (X, d) be a 2-metric space, $f : X \to X$.

If for any $x \in X$, set : $x_0 = x, x_1 = fx_0$, $x_2 = fx_1$, , $x_{n+1} = fx_n$, .., we have :

$$d(x_i, x_j, x_k) = 0 , \forall i, j, k \in I^+ ,$$

then we say that f has property (H) in (X, d) .

Definition 3. Let (X, d) be a 2 metric space , $f , g : X \to X$.

If for any $x \in X$ set : $x_0 = x$, $x_1 = fx_0$, $x_2 = gx_1, \ldots, x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}, \ldots$, we have $d(x_i, x_j, x_k) = 0$, $\forall i, j, k \in I^+$, then we say that f and g have common property (H) in (X, d).

Example. Let $J = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ Define a 2-metric d on J as :

$$d((a_1, a_2), (b_1, b_2), (c_1, c_2)) = \frac{1}{2} \left| \det \begin{pmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{pmatrix} \right| .$$

Define $f : (J, d) \to (J, d)$ as :

$$f((a, b)) = (a^{1/2}, b^{1/2}).$$

Take $x_0 = ((\frac{10}{11})^4, (\frac{11}{12})^4)$, $x_1 = fx_0 = ((\frac{10}{11})^2, (\frac{11}{12})^2)$,

$x_2 = fx_1 = (\frac{10}{11}, \frac{11}{12})$. Then $d(x_0, x_1, x_2) \neq 0$, and f does not have property (H) in (J, d).

Remark 1. Let $x : (x_1, x_2, \ldots, x_n)$, $y : (y_1, y_2, \ldots, y_n)$, and $z : (z_1, z_2, \ldots, z_n)$ be three points in the Euclidean 2-metric space (R^n, d),

$$d(x, y, z) = \frac{1}{2} \left| |\overrightarrow{xy}| \, . \, |\overrightarrow{xz}| \, . \sin \alpha \right| ,$$

where $|\overrightarrow{xy}|$ denotes the length of the vector from x to y, $|\overrightarrow{xz}|$ denotes the length of the vector from x to z, and α is the angle between $\overrightarrow{xy}$ and $\overrightarrow{xz}$.

The following is a special case of Theorem 1 to be stated later.

If $d(x, y, z) = 0$, *then* $|\overrightarrow{xy}| = 0$, *or* $|\overrightarrow{xz}| = 0$, *or* $\alpha = 0$ *or* π, *and* x, y, *and* z *are collinear* .

From the above argument we know : If a mapping f has property (H) in a Euclidean 2-metric space , then the points given by the iteration of f at any point are collinear .

Remark 2 . Later we shall prove that , if a mapping f satisfies the conditions of any fixed point theorem in [2] to [12], then f has property (H) in a 2-metric space, and, if mappings f, g satisfy the conditions of any common fixed point theorem in [2] to [12], then f and g have common property (H) in a 2-metric space . So all the theorems in [2] to [12] fail to discuss the mappings which do not have property (H) in a 2-metric space ,

we are not interested in what the property (H) leads to, since the mappings which have property (H) in a 2-metric are trivial We prove :

Theorem 1. *Let* (x, d) *be a 2-metric space . Mappings* $f, g, : X \to X$ *satisfy the following condition :*

$$d(fx, gy, a) \leqslant \phi(d(x, y, a), d(x, fx, a)\, d(y, gy, a), \quad d(x, gy, a), d(y, fx, a)) \quad \ldots\ldots\ldots\ldots (1^*)$$

for all $x, y, a \in X$, *where the mapping* $\phi : R^5_+ \to R_+$ *is non-decreasing in each variable, and*

$$\phi(t, t, t, t, t) < t, \text{ for each } t > 0 . \quad \ldots\ldots\ldots (2^*)$$

Then f *and* g *have common property* (H) *in* (X, d) .

Proof . First Prove :

$$d(x_{n+2}, x_{n+1}, x_n) = 0 \text{ for all } n \in I^+ \quad \ldots\ldots \qquad (A)$$

When n is even let $n = 2i$, from (1*) we have

$$d(x_{2i+2}, x_{2i+1}, x_{2i}) = d(x_{2i+1}, x_{2i+2}, x_{2i})$$
$$= d(fx_{2i}, gx_{2i+1}, x_{2i})$$

$$\leqslant \phi\,(d(x_{2i}, x_{2i+1}, x_{2i}),\ d(x_{2i}, x_{2i+1}, x_{2i}),$$

$$d(x_{2i+1}, x_{2i+2}, x_{2i}),\ d(x_{2i}, x_{2i+2}, x_{2i}),\ d(x_{2i+1}, x_{2i+1}, x_{2i}))$$

$$= \phi\,(0, 0, d(x_{2i+2}, x_{2i+1}, x_{2i}), 0, 0).$$

Suppose $d(x_{2i+2}, x_{2i+1}, x_{2i}) = t^* > 0$. Since ϕ is nondecreasing by (2*) we have $t^* \leqslant \phi(t^*, t^*, t^*, t^*, t^*) < t^*$, a contradiction. Hence $d(x_{2i+2}, x_{2i+1}, x_{2i}) = 0$. Similaly we can prove $d(x_{2i+3}, x_{2i+2}, x_{2i+1}) = 0$.

We now prove each of the following :

$$d(x_n\ \ x_{n+1}, x_0) = 0, \text{ for all } n \in I^+ \quad \ldots\ldots\ldots \qquad (B)$$

$$d(x_n, x_{n+1}, x_1) = 0, \text{ for all } n \in I^+ \quad \ldots\ldots\ldots \qquad (C)$$

$$d(x_0, x_1, x_k) = 0, \text{ for all } k \in I^+ \quad \ldots\ldots\ldots \qquad (D)$$

$$d(x_n, x_{n+1}, x_k) = 0, \text{ for all } n \in I^+, \text{ any } k \in I^+ \qquad (E)$$

$$d(x_i, x_j, x_k = 0, \text{ for all } i, j, k \in I^+ \quad \ldots\ldots\ldots \qquad (F)$$

Note that (B) is trivially true if n is 0 or 1. Assume the induction hypothesis, suppose n is odd, and let $n = 2m+1$. Using (1*) we have :

$$d(x_{2m+1}, x_{2m+2}, x_0)$$
$$= d(fx_{2m}, gx_{2m+1}, x_0)$$

$$\leqslant \phi\,(\,d\,(\,x_{2m}\,,\,x_{2m+1}\,,\,x_0\,)\,,\;d\,(\,x_{2m}\,,\,x_{2m+1}\,,\,x_0\,)\,,\;d\,(\,x_{2m+1}\,,$$
$$x_{2m+2}\,,\,x_0\,)\,,\,d\,(\,x_{2m}\,,\,x_{2m+2}\,,\,x_0\,)\,,\,d\,(\,x_{2m+1}\,,\,x_{2m+1}\,,\,x_0\,)\,)\,.$$
$$=\phi\,(\,0,\,0,\,d\,(\,x_{2m+1}\,,\,x_{2m+2}\,,\,x_0\,)\,,\,d\,(\,x_{2m}\,,x_{2m+2}\,,\,x_0\,)\,,\,0\,)\,.$$

Using the simplex inequlity and (A), we have :

$$d\,(\,x_{2m}\,,\,x_{2m+2}\,,\,x_0\,)$$
$$\leqslant d\,(\,x_{2m+1},\,x_{2m+2},\,x_0\,)+d\,(\,x_{2m},\,x_{2m+1}\;x_0\,)+d\,(\,x_{2m},\,x_{2m+2},\,x_{2m+1}\,)$$
$$=d\,(\,x_{2m+1},\,x_{2m+2},\,x_0\,)\,.$$

If $d\,(\,x_{2m},\,x_{2m+2},\,x_0\,)\neq 0$, then, from (2*) we obtain a contradiction. The proof under the assumption that n is even is similar, as is the proof of (C) .

(D) is trivially true for $k=0$. Assume the induction hypothesis . From the simplex inequality we have :

$$d\,(\,x_0,\,x_1,\,x_{k+1}\,)$$
$$\leqslant d\,(\,x_k,\,x_1,\,x_{k+1}\,)+d\,(\,x_0,\,x_k,\,x_{k+1}\,)+d\,(\,x_0,\,x_1,\,x_k\,)\,.$$

The first two terms are zero by (B) and (C) and the third is zero by the induction assumption . Hence $d\,(\,x_0,\,x_1,\,x_{k+1}\,)=0$ and (D) is proved .

(E) is true for $n=0$ by (D) . Assume the induction hypothesis. Suppose n is even and let $n=2m$. From (1*) we have :

$$d\,(\,x_{2m},\,x_{2m+1},\,x_k\,)=d\,(\,gx_{2m-1},\,fx_{2m},\,x_k\,)$$
$$\leqslant \phi\,(\,d(\,x_{2m},\,x_{2m-1},\,x_k\,),\,d\,(\,x_{2m},\,x_{2m+1},\,x_k\,)\,,$$
$$d\,(\,x_{2m-1},\,x_{2m},\,x_k\,)\,,\,d\,(\,x_{2m},\,x_{2m},\,x_k\,)\,,$$
$$d\,(\,x_{2m-1},\,x_{2m+1},\,x_k\,)\,)\,.$$

Using the simplex inequality, the induction assumption, and (A), $d(x_{2m-1}, x_{2m+1}, x_k) \leqslant d(x_{2m}, x_{2m+1}, x_k)$. If $d(x_{2m}, x_{2m+1}, x_k) \neq 0$, then (2*) leads to a contradiction. The proof under the assumption that n is odd is similar to the above argument.

To prove (F). we may assume, without loss of generality, that $i \leqslant j$. Moreover, we may assume strict ineqnality, since, otherwise, we obtain zero by the definition of 2-metric spaces. Using the simplex inequality, we have :

$$d(x_i, x_j\ x_k)$$

$$\leqslant d(x_{j-1}, x_j, x_k) + d(x_i, x_{j-1}, x_k) + d(x_i, x_j, x_{j-1})$$

$$= d(x_i, x_{j-1}\ x_k),$$

since the other two terms are zero by (E)

Continuing in this same manner, we obtain $d(x_i, x_{j-1}, x_k) \leqslant d(x_i, x_{j-2}, x_k) \leqslant \ldots \leqslant d(x_i, x_{i+1}, x_k) = 0$ by (E).

A different proof of Theorem 1 appears in the appendix. It is included for its value as a proof technique.

There is a special case of Theorem 1; that is :

Theorem 2. *Let f be a self-mapping on a 2-metric space (X, d) such that :*

$$d(fx, fy, a) \leqslant \phi(d(x, y, a), d(x, fx, a), d(y, fy, a),$$
$$d(x, fy, a), d(y, fx, a)) \qquad (3^*)$$

for all $x, y, a \in X$, ϕ satisfies (2) in Theorem 1. Then f has property (H) in (X, d).*

Remark 3. It is easy to check that all the contractive type mappings in the theorems in [5], [8], [9], [10] and [12] have property (H) in (X, d)

or common property (H) in (X, d) , since they all are special cases of (1*) in Theorem 1 or (3*) in Theorem 2 .

3. MAIN RESULTS (PART II)

Lemma 1 formalizes the fact that condition (E) implies (F) .

Lemma 1. *Let* $\{ x_n \}$ *be a sequence in 2-metric space* (X, d) *and*

$$d (x_m, x_{m+1}, x_n) = 0, \ \forall m, n \in I^+ \quad . \quad \ldots\ldots\ldots\ldots \tag{1}$$

Then $d (x_i , x_j , x_k) = 0, \forall i, j, k \in I^+$.

Theorem 3. *Let* $\{ x_n \}$ *be a sequence in* **2**-*metric space* (X, d) *If there exists a real number* $k \in [0, 1]$ *such that, for all* $n \in N$ *and all* $a \in X$,

$$d (x_n, x_{n+1}, a) \leqslant k \,.\, d (x_{n-1}, x_n, a) , \quad \ldots\ldots\ldots \tag{2}$$

then

$$d (x_i , x_j , x_k) = 0, \ \forall i , j, k \in I^+ .$$

Proof. From (2) we know :

$$d (x_n, x_{n+1}, x_m) \leqslant k \,.\, d (x_{n-1}, x_n, x_m) ,$$

for all $n \in N$ and $m \in I^+$.

1^0 When $m = n$ or $n + 1$, $d (x_n, x_{n+1}, x_m) = 0$

2^0 When $m < n$,

$$\begin{aligned} d (x_n, x_{n+1} , x_m) &\leqslant k \,.\, d (x_{n-1} , x_n , x_m) \\ &\leqslant k^2 \,.\, d (x_{n-2} , x_{n-1} , x_m) \\ &\vdots \\ &\leqslant k^{n-m} \,.\, d (x_m , x_{m+1} , x_m) = 0 \end{aligned}$$

3^0 When $m > n + 1$,

$$d(x_n, x_{n+1}, x_m)$$

$$\leqslant d(x_{m-1}, x_{n+1}, x_m) + d(x_n, x_{m-1}, x_m) + d(x_n, x_{n+1}, x_{m-1}).$$

By 1^0 and 2^0 we have $d(x_{m-1}, x_m, x_{n+1}) = 0$ and

$d(x_{m-1}, x_m, x_n) = 0$. Then $d(x_n, x_{n+1}, x_m \leqslant d(x_n, x_{n+1}, x_{m-1})$.

Continuing in this way we have :

$$\begin{aligned} d(x_n, x_{n+1}, x_m) &\leqslant d(x_n, x_{n+1}, x_{m-1}) \\ &\leqslant d(x_n, x_{n+1}, x_{m-2}) \\ &\vdots \\ &\leqslant d(x_n, x_{n+1}, x_{n+1}) = 0. \end{aligned}$$

4^0 When $n = 0$,

$$(x_0, x_1, x_m) \leqslant d(x_2. x_m, x_0) + d(x_1, x_2, x_0) + d(x_1, x_2, x_m).$$

By 1^0, 2^0, 3^0 we have $d(x_1, x_2. x_0 = 0$ and

$d(x_1, x_2, x_m) = 0$ Then $d(x_0, x_1, x_n) \leqslant d(x_2. x_m, x_0)$.

Continuing in this way we have :

$$\begin{aligned} d(x_0, x_1, x_m) &\leqslant d(x_2, x_m, x_0) \\ &\leqslant d(x_3, x_n, x_0) \\ &\vdots \\ &\leqslant d(x_{n-1}, x_n, x_0) = 0. \end{aligned}$$

From 1^0, 2^0, 3^0 and 4^0 we have : $d(x_n, x_{n+1}, x_m) = 0$,

$\forall m,n \in I^+$. Then, by Lemma 1, we have : $d(x_i, x_j, x_k) = 0$, $\forall i, j, k \in I^+$.

Remark 4. By Lemma 1 and Theorem 3, it is easy to check that all the contractive type mappings in the theorems in [4] and [11] have property (H) in (X,d) or common property (H) in (X, d).

4. MAIN RESULTS (PART III)

In this part we shall illustrate that all the proofs of the theorems in [2], [3], [6], and [7] are incomplete or wrong.

Following [1], we have Definitions 4 and 5.

Definition 4. A sequence $\{x_n\}$ in a 2-metric space (X, d) is convergent and $x \in X$ is the limit of this sequence if

$$\lim_{n\to\infty} d(x_n, x, a) = 0 \quad \text{for all } a \in X.$$

Definition 5. A sequence $\{x_n\}$ in a 2-metric space (X, d) is called a Cauchy sequence if $\lim_{m,n\to\infty} d(x_m, x_n, a) = 0$, for all $a \in X$.

Remark 5. From Definitions 4 and 5, we know: In the proof of the Theorem of [2], the author needs to prove that the inequality $\rho(x_n, x_{n+1}, a) \leqslant q\, \rho(x_{n-1}, x_n, a)$ is true for all points a in X. It is only shown to be true for some points a of X. Since the authors of [3], [6] and [7] use the same proof technique, those results are also incorrect.

We are not interested in whether the statement

"$\rho(x_n ; x_{n+1}, a) \leqslant q\, \rho(x_{n-1}, x_n, a)$" is true for all $a \in X$ or not, since by Theorem 3 we know that:

$$\text{"}\rho(x_n, x_{n+1}, a) \leqslant q\, \rho(x_{n-1}, x_n, a), \quad \textit{for all } a \in X \textit{ and all } a \in N\text{"}$$

implies that: $\rho(x_i, x_j, x_k) = 0$, *for all* $i, j, k \in I^+$.

Thus T has property (H) in (X, d)

5. CONCLUSIONS

In this article we have shown that all the theorems in [2] to [12] only can discuss mappings that have property (H) in a 2-metric space.

If f has property (H) in Euclidean 2-metric space (R^2, A), where $A(x, y, z)$ denotes the area of the triangle $\triangle xyz$ formed by joining the three points x, y, z in R^2. Then for each x in R^2, the iteration of f at x is in a straight line. So f is a trivial mapping. In fact, the definition of a 2-metric space given by Gähler is not strong. The Euclidean 2-metric space (R^2, A) has the following property denoted by property (H^*) :

"*If* $A(a, b, x) = 0 = A(a, b, y)$ *and* $A(a, x, y) \neq 0$, *then* $a = b$"

But, if we define a new 2-metric A^* on R^2 by $A^*(x, y, z) =$ the area of the intersection of $\triangle xyz$ and the unit disc in R^2, then we have :

$$A^*((2, 0), (-2, 9), (2, 9)) = 0$$

$$A^*((-2, 0), (-2, 9), (2, 9)) = 0 \text{ and}$$

$$A^*((-2, 0), (2, 0), (2, 9)) \neq 0, \text{ but}$$

$$(2, 9) \neq (-2, 9).$$

Hence (R^2, A^*) doesn't have property (H^*). Thus not all 2-metric spaces have property (H^*) as Euclidean 2-metric spaces have. Moreover, in a 2-metric space (X, d) if we want to prove "$a = b$", we must prove "$d(a, b, x) = 0$, for all x in (X, d)".

We cannot establish a good fixed point theorem in a 2-metric space by considering iterations of a contractive type mapping. We suggest that we investigate 2-metric spaces that have property H^*. Then we have a better chance of establishing a useful fixed point theorem in 2 metric spaces.

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APPENDIX

The following is a different way of proving Theorem 1.

Proof. First prove :

$$d\,(x_{n+2}, x_{n+1}, x_n) = 0 \text{ for all } n \in I^+ \quad \dots \text{(A)}$$

When n is even let $n = 2i$, from (1*) we have

$$d\,(x_{2i+2}, x_{2i+1}, x_{2i}) = d\,(x_{2i+1}, x_{2i+2}, x_{2i})$$

$$= d\,(fx_{2i}, gx_{2i+1}, x_{2i})$$

$$\leqslant \phi\,(d\,(x_{2i+2}, x_{2i+1}, x_{2i}), d\,(x_{2i}, x_{2i+1}, x_{2i}),$$

$$d\,(x_{2i+1}, x_{2i+2}, x_{2i}), d\,(x_{2i}, x_{2i+2}, x_{2i}), d\,(x_{2i+1}, x_{2i+1}, x_{2i}))$$

$$= \phi\,(0, 0, d\,(x_{2i+2}, x_{2i+1}, x_{2i}), 0, 0).$$

Suppose $d\,(x_{2i+2}, x_{2i+1}, x_{2i}) = t^* > 0$. Since ϕ is nondecreasing by (2*) we have $t^* \leqslant \phi\,(t^*, t^*, t^*, t^*, t^*) < t^*$, a contradiction. Hence $d\,(x_{2i+2}, x_{2i+1}, x_{2i}) = 0$. Similarly, we can prove $d\,(x_{2i+3}, x_{2i+2}, x_{2i+1}) = 0$. We now show that :

$$d\,(x_{n+3}, x_{n+1}, x_n) = d\,(x_{n+3}, x_{n+2}, x_n) = 0$$

$$\text{for all } n \in I^+. \quad \dots \quad \text{(B)}$$

By the simplex inequality :

$$d\,(x_{n+3}, x_{n+1}, x_n)$$

$$\leqslant d\,(x_{n+2}, x_{n+1}, x_n) + d\,(x_{n+3}, x_{n+2}, x_n) + d\,(x_{n+3}, x_{n+1}, x_{n+2})$$

$$d\,(x_{n+3}, x_{n+2}, x_n)$$

$$\leqslant d\,(x_{n+1}, x_{n+2}, x_n) + d\,(x_{n+3}, x_{n+1}, x_n) + d\,(x_{n+3}, x_{n+2}, x_{n+1}).$$

From (A) we have

$$d(x_{n+3}, x_{n+1}, x_n) \leqslant d(x_{n+3}, x_{n+2}, x_n),$$

and

$$d(x_{n+3}, x_{n+2}, x_n) \leqslant d(x_{n+3}, x_{n+1}, x_n),$$

then $d(x_{n+3}, x_{n+2}, x_n) = d(x_{n+3}, x_{n+1}, x_n)$. When n is even, let $n = 2i$. Then

$$d(x_{2i+3}, x_{2i+1}, x_{2i}) = d(x_{2i+3}, x_{2i+2}, x_{2i})$$

$$= d(fx_{2i+2}, gx_{2i+1}, x_{2i})$$

$$\leqslant \phi(d(x_{2i+2}, x_{2i+1}, x_{2i}),\ d(x_{2i+2}, x_{2i+3}, x_{2i}).$$

$$d(x_{2i+1}, x_{2i+2}, x_{2i}), (x_{2i+2}, x_{2i+2}, x_{2i}),$$

$$d(x_{2i+1}, x_{2i+3}, x_{2i}))$$

$$= \phi(0, d(x_{2i+3}, x_{2i+2}, x_{2i}), 0, 0, d(x_{2i+3}, x_{2i+1}, x_{2i})).$$

Suppose $d(x_{2i+3}, x_{2i+2}, x_{2i}) = d(x_{2i+3}, x_{2i+1}, x_{2i}) = t^* > 0$, then by (2*), $t^* \leqslant \phi(t^*, t^*, t^*, t^*, t^*) < t^*$, a contradiction. This implies that (B) is true when n is even.

Similarly, we can show that (B) is true when n is odd. We now show that $d(x_i, x_j, x_k) = 0$, for all $i, j, k \in I^+$.

[Step 1]. $d(x_i, x_j, x_k) = 0, \forall i, j, k \in \{0, 1, 2, 3\}$.

Since (A) is true, we only have to show

$$(1^0) \qquad \begin{cases} d(x_3, x_1, x_0) = 0 \\ d(x_3, x_2, x_0) = 0. \end{cases}$$

Since (B) is true, (1^0) is trivial.

[Step 2]. $d(x_i, x_j, x_k) = 0, \forall i, j, k \in \{0, 1, 2, 3, 4\}$.

By [Step 1] and (A) we only have to show

$$(2^0)\begin{cases} d(x_4, x_1, x_0) = 0 \\ d(x_4, x_2, x_0) = 0, \\ d(x_4, x_3, x_0) = 0 \end{cases} \qquad (1^1)\begin{cases} d(x_4, x_2, x_1) = 0 \\ d(x_4, x_3, x_1) = 0 \end{cases}$$

(1^1) is trivial since (B) is always true .

$$d(x_4, x_1, x_0) \leqslant d(x_l, x_1, x_0) + d(x_4, x_l, x_0)$$

$$+ d(x_4, x_l, x_1) \quad \text{for} \quad l = 2, 3 .$$

From (1^1) $d(x_4, x_l, x_1) = 0$, $l = 2, 3$.

From [Step 1] $d(x_l, x_1, x_0) = 0$, $l = 2, 3$.

Hence

$d(x_4, x_1, x_0) \leqslant d(x_4, x_l, x_0)$, $l = 2, 3$.

Similarly

$d(x_4, x_l, x_0) \leqslant d(x_4, x_1, x_0$, $l = 2, 3$.

$$\Rightarrow \qquad d(x_4, x_1, x_0) = d(x_4, x_l, x_0), \; l = 2, 3 .$$

Since for all $i, j, \in \{0, 1, 2, 3, 4\}$ either $d(x_i, x_j, x_0) = 0$ or $d(x_i, x_j, x_0)$ appears in (2^0), and

$d(x_i, x_j, x_0) = d(x_4, x_1, x_0)$,

then by (1^*) and (2^*) it's easy to show that :

$$d(x_4, x_1, x_0) = d(x_4, x_l, x_0) = 0, \quad l = 2, 3 .$$

[Step 3]. $d(x_i, x_j ; x_k) = 0$, $\forall i, j, k \in \{0, 1, 2, 3, 4, 5\}$

We only have to show :

$$(3^0)\begin{cases} d(x_5, x_1, x_0) = 0 \\ d(x_5, x_2, x_0) = 0 \\ d(x_5, x_3, x_0) = 0 \\ d(x_5, x_4, x_0) = 0 \end{cases} \qquad (2^1)\begin{cases} d(x_5, x_2, x_1) = 0 \\ d(x_5, x_3, x_1) = 0 \\ d(x_5, x_4, x_1) = 0 \end{cases}$$

$$(1^2)\begin{cases} d(x_5, x_3, x_2) = 0 \\ d(x_5, x_4, x_2) = 0 . \end{cases}$$

The proof of (1^2) and (2^1) is similar to [Step 2] ; that is , "rewrite [Step 2] by considering $d(x_i, x_j, x_k)$, $\forall i, j, k \in \{1, 2, 3, 4, 5\}$ instead of considering $d(x_i, x_j, x_k)$, $\forall i, j, k \in \{0, 1, 2, 3, 4\}$", since (B) is always true .

So we only have to show (3^0)

$$d(x_5, x_1, x_0)$$

$$\leqslant d(x_l, x_1, x_0) + d(x_5, x_l, x_0) + d(x_5, x_1, x_l)$$

$$\rightarrow d(x_5, x_1, x_0) \leqslant d(x_5, x_l, x_0), \quad \text{for } l = 2, 3, 4 .$$

$$d(x_5, x_l, x_0)$$

$$\leqslant d(x_1, x_l, x_0) + d(x_5, x_1, x_0) + d(x_5, x_l, x_1)$$

$$\Rightarrow d(x_5, x_l, x_0) \leqslant d(x_5, x_1, x_0), \text{ for } l = 2, 3, 4 .$$

Hence $d(x_5, x_1, x_0) = d(x_5, x_l, x_0)$, $t = 2, 3, 4$. Since for all $i, j. \in \{0, 1, 2, 3, 4, 5\}$ either $d(x_i, x_j, x_0) = 0$ or $d(x_i, x_j, x_0)$ appears in (3^0), and $d(x_i, x_j, x_0) = d(x_5, x_1, x_0)$, then by (1^*) and (2^*) it's easy to show that :

$d(x_5, x_1, x_0) = d(x_5, x_l, x_0)$, $l = 2, 3, 4$, Suppose continue in this way we can prove :

[Step m-2] , $d(x_i, x_j, x_k) = 0$, $\forall i, j, k, \in \{0, 1, 2, \ldots, m\}$.

To show :

[Step m-1] , $d(x_i, x_j, x_k) = 0$, $\forall i, j, k \in \{0, 1, 2, \ldots, m.$

$m+1\}$. We only have to show

$$[(m-1)^0] \quad \begin{cases} d(x_{m+1}, x_1, x_0) = 0 \\ d(x_{m+1}, x_2, x_0) = 0 \\ \quad \vdots \\ d(x_{m+1}, x_m, x_0) = 0 \end{cases} \quad \cdots$$

$$[(m-2)^1] \quad \begin{cases} d(x_{m+1}, x_2, x_1) = 0 \\ \quad \vdots \\ d(x_{m+1}, x_m, x_1) = 0 \end{cases} \quad \cdots$$

$$\ldots, [1^{m-2}] \begin{cases} d(x_{m+1}, x_{m-1}, x_{m-2}) = 0 \\ d(x_{m+1}, x_m, x_{m-1}) = 0 \end{cases}.$$

Since the proof of $[(m-2)^1]$, $[(m-3)^2]$, ... $[2^{m-3}]$, $[1^{m-2}]$ is similar to [Step $m-2$] , we only have to show $[(m-1)^0]$.
By simplex inequailty we can show that :

$d(x_{m+1}, x_1, x_0) = d(x_{m+1}, x_l, x_0)$, $l = 2, 3, \ldots, m$, then by (1*) , (2*) we can prove $[(m-1)^0]$. By mathematical induction for m the proof is completed .

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